# 2-Groups, 2-Characters, and Burnside Rings[☆]

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**Abstract**

We study 2-representations, i.e., actions of 2-groups on 2-vector spaces. Our main focus is character theory for 2-representations. To this end we employ the technique of extended Burnside rings. Our main theorem is that the Ganter-Kapranov 2-character is a particular mark homomorphism of the Burnside ring. As an application we give a new proof of Osorno's formula for the Ganter-Kapranov 2-character of a finite group.

*Keywords:* 2-group, crossed module, Burnside ring

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This paper develops the theory of 2-representations and their associated 2-characters. A finite group $G$ can act on a 2-vector space, in a sense of Kapranov and Voevodsky [11]. Ganter and Kapranov use 2-traces to associate a 2-character to such a 2-representation [8]. Osorno gives an explicit formula for this 2-character in terms of cohomological data [16].

We would like to contemplate this 2-character and its formula in this paper. We approach it via generalised Burnside rings, building on the work by Gunnells, Rose and the first author [9]. Our main result is an explicit expression for the Ganter-Kapranov character as a mark homomorphism (Theorem 5.2).

We work in a slightly bigger generality as it has little extra cost and could prove useful: instead of a finite group $G$ we work with a 2-group $\mathcal{G} = \widetilde{\mathcal{K}}$ arising from a crossed module $\mathcal{K}$ whose fundamental group $\pi_1(\mathcal{G})$ is finite. Let us explain how this paper is organised.

In Section 1 we set out the terminology of 2-categories, in particular, we define 2-groups, 2-vector spaces, 2-representations and 2-modules. At this stage one can think of a 2-representation as a "semisimple" 2-module. Most of this chapter is well known, yet it is essential to establish our notation and our terminology.

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One important result in this section is Theorem 1.2, an equivalence of two realizations of the 2-category of 2-vector spaces. It is probably known to the experts but we could not find it in the literature.

In Section 2 we roll out our philosophy: a 2-module $\Theta_A$ of $\mathcal{G}$ arises from an action of the crossed module $\mathcal{K}$-action on an algebra $A$ (Proposition 2.3). There is a subtlety: we need to distinguish strict and weak 2-actions. A weak 2-action is a weak homomorphism of crossed modules from $\mathcal{K}$ to the crossed module of 2-automorphisms of $A$. We finish the section with a Morita theorem for strict 2-modules (Theorem 2.6): the 2-modules $\Theta_A$ and $\Theta_B$ are equivalent if and only if $A$ and $B$ are $\mathcal{K}$-Morita equivalent.

Section 3 is devoted to 2-representations. If $\mathcal{K} = (H \to G)$ then a 2-representation of $\mathcal{G}$ yields a 2-cocycle for $G$ (Lemma 3.2). At this stage we uncover another subtlety: the cocycle may or may not be realized on a finite dimensional projective representation of $G$. In the former case we call the cocycle *realizable*. Realizable cocycles lead to split semisimple algebras, while non-realizable cocycles lead to direct sums of full matrix algebras, some of whom are infinite dimensional. We call such algebras *semimatrix*. A semimatrix algebra $A$ with an action of $\mathcal{K}$ gives a 2-representation $\Theta_A^\circ$ of $\mathcal{G}$, equal to $\Theta_A$ if $A$ is semisimple. We can manage not only $\Theta_A$ but also $\Theta_A^\circ$ with some additional care (cf. Corollary 3.3 and Corollary 3.5). The upshot of this chapter is that weak non-unital 2-representations come from strict $\mathcal{K}$-algebras, in particular, are equivalent to strict 2-representations. This gives a convenient insight into a structure of 2-representations. We finish the section with a structure theorem for 2-representations (Theorem 3.7).

In Section 4 we utilise generalised Burnside rings [9] to describe the Grothendieck group of 2-representations of $\mathcal{G}$. It is curious that the Burnside ring is slightly unusual: two elements $\mathbf{x}, \mathbf{y} \in G$ define the same conjugation if $\mathbf{x}\mathbf{y}^{-1}$ is in the centre but may determine different pull-backs of 2-representations.

In Section 5 we define the Ganter-Kapranov 2-character for 2-Groups. We express these 2-characters in terms of the generalised Burnside rings, more precisely, the Ganter-Kapranov 2-character is a certain mark homomorphism (Theorem 5.2). This result is the main theorem of this paper.

Starting from Section 6, we work with a group, as a particular example of a 2-group. In Section 6 we make some technical preparations, namely we write an explicit formula for Shapiro isomorphism on the level of cocycles (Theorem 6.1). It may be known to experts but we could not find it in the literature. It gives a slightly stronger version of Shapiro isomorphism: two complexes are not only quasiisomorphic but also homotopically equivalent.

In Section 7 we explicitly calculate the Ganter-Kapranov 2-character for the groups. Our first result (Theorem 7.2) is a formula that follows immediately from mark homomorphisms. The second result is a known formula by Osorno (Theorem 7.5). We intentionally prove the known formula as well to demonstrate the power of our method. We finish the section with two conjectures suggesting how to generalise the content of this section to the 2-groups.

In the final Section 8 we restate two famous conjectures on the level of 2-representations: Lusztig's

Conjecture about base sets for the double cells and the McKay Conjecture for the number of $p'$-characters of a finite group.

## 1. Introduction to 2-representations

Let us clarify what we mean by 2-representations of 2-groups in this paper. They have been studied by several authors [1, 2, 3, 7, 8, 16]. In general, we follow the terminology of Benabou, who distinguishes 2-categories and bicategories [4]. Both structures consist of a class $\mathcal{C}_0$ of 0-objects, a category $\mathcal{C}_1(x,y)$ for each pair of 0-objects, a unit 1-object (or 1-morphism - these are synonyms) $\mathbf{i}_x \in \mathcal{C}_1(x,x)$ and composition bifunctors

$$\diamond = \diamondsuit_{x,y,z} : \mathcal{C}_1(x,y) \times \mathcal{C}_1(y,z) \to \mathcal{C}_1(x,z).$$

In particular, $\mathcal{C}_2(u,v)$ is the set of 2-morphism between 1-objects (a.k.a, 1-morphisms) $u$ and $v$ of $\mathcal{C}_1(x,y)$. We use two symbols for compositions. The circle $\circ$ stands for the usual compositions of morphisms or 2-morphism that we write right-to-left (or bottom-to-top in globular notation). The diamond $\diamond$ stands for the composition bifunctor in a bicategory that we write left-to-right. In a 2-category the compositions of 1-morphisms are associative and unital. Let $\mathcal{I}_{x,y}$ be the identity endofunctor on $\mathcal{C}_1(x,y)$. In a bicategory associativity is a family of natural isomorphisms of trifunctors

$$\mathrm{Ass}_{w,x,y,z} : \diamondsuit_{w,x,z} \circ (\mathcal{I}_{w,x} \times \diamondsuit_{x,y,z}) \Rightarrow \diamondsuit_{w,y,z} \circ (\diamondsuit_{w,x,y} \times \mathcal{I}_{y,z}),$$

$$\diamondsuit_{w,x,z} \circ (\mathcal{I}_{w,x} \times \diamondsuit_{x,y,z}),\ \diamondsuit_{w,y,z} \circ (\diamondsuit_{w,x,y} \times \mathcal{I}_{y,z}) : \mathcal{C}_1(w,x) \times \mathcal{C}_1(x,y) \times \mathcal{C}_1(y,z) \to \mathcal{C}_1(w,z)$$

such that the pentagon diagrams are commutative. Similarly, unitality in a bicategory is two families of natural isomorphisms of functors

$$\mathrm{RUn}_{x,y} : \diamondsuit_{x,y,y}(\ \ , \mathbf{i}_y) \Rightarrow \mathcal{I}_{x,y}, \quad \mathrm{LUn}_{x,y} : \diamondsuit_{x,x,y}(\mathbf{i}_x, \ \ ) \Rightarrow \mathcal{I}_{x,y}$$

such that the triangle diagrams are commutative. A bicategory is *small* if it consists of sets on each level: $\mathcal{C}_0$ is a set and all categories $\mathcal{C}_1(x,y)$ are small. We will have both 2-categories and bicategories in this paper.

*A (weak) 2-functor* (between bicategories) $F : \mathcal{C} \to \mathcal{D}$ is a datum

$$F^0 : \mathcal{C}_0 \to \mathcal{D}_0, \quad F^1_{x,y} : \mathcal{C}_1(x,y) \to \mathcal{D}_1(F^0(x), F^0(y)), \quad F^2_x : \mathbf{i}_{F^0(x)} \Rightarrow F^1_{x,x}(\mathbf{i}_x),$$

$$F^2_{x,y,z} : \diamondsuit_{F^0(x),F^0(y),F^0(z)} \circ (F^1_{x,y} \times F^1_{y,z}) \Rightarrow F^1_{x,z} \circ \diamondsuit_{x,y,z}$$

where $F^0$ is a function, $F^1_{x,y}$ is a family of functors, $F^2_x$ is a family of 2-isomorphisms (i.e., 2-morphisms that are isomorphisms), and $F^2_{x,y,z}$ is a family of compatibility conditions, natural isomorphisms of bifunctors from $\mathcal{C}_1(x,y) \times \mathcal{C}_1(y,z)$ to $\mathcal{D}_1(F^0(x), F^0(z))$. The requirement is that the hexagon and the square diagrams are commutative (cf. a definition of monoidal functor). The hexagon diagram ensures that the two possible natural transformations

$$nt_1, nt_2 : \diamondsuit_{F^0(w),F^0(y),F^0(z)} \circ (\diamondsuit_{F^0(w),F^0(x),F^0(y)} \times \mathrm{Id}) \circ (F^1_{w,x} \times F^1_{x,y} \times F^1_{y,z}) \Rightarrow F^1_{w,z} \circ \diamondsuit_{w,x,z} \circ (\mathrm{Id} \times \diamondsuit_{x,y,z})$$

of trifunctors from $\mathcal{C}_1(w,x) \times \mathcal{C}_1(x,y) \times \mathcal{C}_1(y,z)$ to $\mathcal{D}_1(F^0(w), F^0(z))$ are equal. Similarly, the two square diagrams equate possible natural transformations

$$\diamondsuit_{F^0(x),F^0(y),F^0(y)} \circ (F^1_{x,y} \times \mathbf{i}_{F^0(y)}) \Rightarrow F^1_{x,y} \circ \diamondsuit_{x,y,y} \circ (\mathrm{Id} \times \mathbf{i}_y), \ \ \diamondsuit_{F^0(x),F^0(x),F^0(y)} \circ (\mathbf{i}_{F^0(x)} \times F^1_{x,y}) \Rightarrow F^1_{x,y} \circ \diamondsuit_{x,x,y} \circ (\mathbf{i}_x \times \mathrm{Id})$$

of functors from $\mathcal{C}_1(x,y)$ to $\mathcal{D}_1(F^0(x), F^0(y))$. We say that $F$ is *unital* if each $F^2_x$ is the identity. We say that $F$ is *strict* if all $F^2_x$ and $F^2_{x,y,z}$ are the identities.

*A natural 2-transformation* $\psi : F \Rightarrow G$ between 2-functors $F, G : \mathcal{C} \to \mathcal{D}$ is a datum

$$\psi^1_x \in \mathcal{D}_1(F^0(x), G^0(x)), \quad \psi^2_{x,y} : F^1_{x,y} \diamond \psi^1_y \Rightarrow \psi^1_x \diamond G^1_{x,y}$$

where $\psi^1$ is a family of 1-morphisms in $\mathcal{D}$ and $\psi^2$ is a family of natural transformations of functors $\mathcal{C}_1(x,y) \to \mathcal{D}_1(F^0(x), G^0(y))$ satisfying two coherence conditions that we will spell out following Barrett and Mackaay [3]. The first condition is that

$$\psi_{\mathbf{a},\mathbf{b}} = \psi_{\mathbf{a}} \circ \psi_{\mathbf{b}}$$

where these three 2-morphisms are defined for any pair of 1-objects $\mathbf{a} \in \mathcal{C}_1(x,y)$, $\mathbf{b} \in \mathcal{C}_1(y,z)$ by

$$\psi_{\mathbf{a},\mathbf{b}} : F^1_{x,y}(\mathbf{a}) \diamond F^1_{y,z}(\mathbf{b}) \diamond \psi^1_z \xrightarrow{F^2_{x,y,z}(\mathbf{a},\mathbf{b})} F^1_{x,z}(\mathbf{a} \diamond \mathbf{b}) \diamond \psi^1_z \xrightarrow{\psi^2_{x,z}(\mathbf{a} \diamond \mathbf{b})} \psi^1_x \diamond G^1_{x,z}(\mathbf{a} \diamond \mathbf{b}) \xrightarrow{G^2_{x,y,z}(\mathbf{a},\mathbf{b})^{-1}} \psi^1_x \diamond G^1_{x,y}(\mathbf{a}) \diamond G^1_{y,z}(\mathbf{b}),$$

$$\psi_{\mathbf{b}} : F^1_{x,y}(\mathbf{a}) \diamond F^1_{y,z}(\mathbf{b}) \diamond \psi^1_z \xrightarrow{\mathrm{Id}_{F^1_{x,y}(\mathbf{a})} \diamond \psi^2_{y,z}(\mathbf{b})} F^1_{x,y}(\mathbf{a}) \diamond \psi^1_y \diamond G^1_{y,z}(\mathbf{b}),$$

$$\psi_{\mathbf{a}} : F^1_{x,y}(\mathbf{a}) \diamond \psi^1_y \diamond G^1_{y,z}(\mathbf{b}) \xrightarrow{\psi^2_{x,y}(\mathbf{a}) \diamond \mathrm{Id}_{G^1_{y,z}(\mathbf{b})}} \psi^1_x \diamond G^1_{x,y}(\mathbf{a}) \diamond G^1_{y,z}(\mathbf{b}).$$

The second condition is that the 2-morphism

$$\psi^1_x \xrightarrow{LUn^{-1}} \mathbf{i}_{F^0(x)} \diamond \psi^1_x \xrightarrow{F^{1\ -1}_x} F^1_{x,x}(\mathbf{i}_x) \diamond \psi^1_x \xrightarrow{\psi^2_{x,x}(\mathbf{i}_x)} \psi^1_x \diamond G^1_{x,x}(\mathbf{i}_x) \xrightarrow{G^1_x} \psi^1_x \diamond \mathbf{i}_{G^0(x)} \xrightarrow{RUn} \psi^1_x$$

is equal to $\mathrm{Id}_{\psi^1_x}$ for each $x \in \mathcal{C}_0$. If all $\psi^1_x$ are 1-equivalences and all $\psi^2_{x,y}$ are natural isomorphisms, we say that $\psi$ is *a natural 2-isomorphism*. Recall that a 1-morphism $\mathbf{a} \in \mathcal{C}_1(x,y)$ is a 1-equivalence if it is quasiinvertible, i.e., there exists $\mathbf{a}^{-1} \in \mathcal{C}_1(y,x)$ such that $\mathbf{a} \diamond \mathbf{a}^{-1}$ is isomorphic to $\mathbf{i}_x$ and $\mathbf{a}^{-1} \diamond \mathbf{a}$ is isomorphic to $\mathbf{i}_y$. It is a 1-isomorphism if it is invertible, i.e., there exists $\mathbf{a}^{-1} \in \mathcal{C}_1(y,x)$ such that $\mathbf{a} \diamond \mathbf{a}^{-1} = \mathbf{i}_x$ and $\mathbf{a}^{-1} \diamond \mathbf{a} = \mathbf{i}_y$.

Two bicategories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there exist *a 2-equivalence*, i.e., a 2-functor $F : \mathcal{C} \to \mathcal{D}$ that admits a quasiinverse 2-functor $G : \mathcal{D} \to \mathcal{C}$ with natural 2-isomorphisms $FG \xrightarrow{\cong} \mathcal{I}d_{\mathcal{D}}$ and $GF \xrightarrow{\cong} \mathcal{I}d_{\mathcal{C}}$.

*A 2-group* $\mathcal{G}$ is a bicategory such that $\mathcal{G}_0$ is a one-element set, each 2-morphism is a 2-isomorphism, and each 1-morphism is a 1-equivalence. *A strict 2-group* $\mathcal{G}$ is a 2-category such that $\mathcal{G}_0$ is a one-element set, each 2-morphism is a 2-isomorphism, and each 1-morphism is a 1-isomorphism.

*A homomorphism* of 2-groups is a 2-functor $F : \mathcal{G} \to \mathcal{H}$. It could be strict or unital, if such is the 2-functor. The 2-groups $\mathcal{G}$ and $\mathcal{H}$ are *equivalent* if there is a 2-equivalence $F : \mathcal{G} \to \mathcal{H}$.

Strict small 2-groups are equivalent to 2-groups that arise from crossed modules [1]. Let $\mathcal{K} = (H \xrightarrow{\partial} G)$ be a crossed module. We assume that $G$ acts on $H$ on the left:

$$\mathbf{g} : h \mapsto {}^{\mathbf{g}}h.$$

Its fundamental groups are $\pi_2(\mathcal{K}) = \ker(\partial)$ and $\pi_1(\mathcal{K}) = \operatorname{coker}(\partial)$. The crossed module $\mathcal{K}$ determines a small strict 2-group $\widetilde{\mathcal{K}}$:

1. $\widetilde{\mathcal{K}}_0$ is a one point set $\{\star\}$.
2. The objects of the category $\widetilde{\mathcal{K}}_1(\star,\star)$ is the group $G$.
3. For a pair $\mathbf{g}_1, \mathbf{g}_2 \in G$ the 2-morphisms in the category $\widetilde{\mathcal{K}}_1(\star,\star)$ are

$$\widetilde{\mathcal{K}}_2(\mathbf{g}_1,\mathbf{g}_2) := \{\mathbf{g}_1 \xRightarrow{h} \mathbf{g}_2\} \;=\; \{\; \begin{tikzcd} \star \arrow[r, bend left=50, "\mathbf{g}_2"] \arrow[r, bend right=50, "\mathbf{g}_1"'] \arrow[r, phantom, "\Uparrow h"] & \star \end{tikzcd} \;\mid\; h \in H,\; \mathbf{g}_2 = \partial(h)\mathbf{g}_1\},$$

while their composition is determined by the product in $H$:

$$[\mathbf{g}_2 \xRightarrow{h_2} \mathbf{g}_3] \circ [\mathbf{g}_1 \xRightarrow{h_1} \mathbf{g}_2] = \begin{tikzcd} \star \arrow[r, bend left=60, "\mathbf{g}_3"] \arrow[r, "\mathbf{g}_2" description] \arrow[r, bend right=60, "\mathbf{g}_1"'] \arrow[r, phantom, shift left=3ex, "\Uparrow h_2"] \arrow[r, phantom, shift right=3ex, "\Uparrow h_1"] & \star \end{tikzcd} = \begin{tikzcd} \star \arrow[r, bend left=50, "\mathbf{g}_3"] \arrow[r, bend right=50, "\mathbf{g}_1"'] \arrow[r, phantom, "\Uparrow h_2h_1"] & \star \end{tikzcd}.$$

4. The composition bifunctor comes from the action and the multiplication:

$$[\mathbf{f} \xRightarrow{h} \mathbf{g}] \diamond [\mathbf{f}_1 \xRightarrow{h_1} \mathbf{g}_1] = \begin{tikzcd} \star \arrow[r, bend left=50, "\mathbf{g}"] \arrow[r, bend right=50, "\mathbf{f}"'] \arrow[r, phantom, "\Uparrow h"] & \star \arrow[r, bend left=50, "\mathbf{g}_1"] \arrow[r, bend right=50, "\mathbf{f}_1"'] \arrow[r, phantom, "\Uparrow h_1"] & \star \end{tikzcd} = \begin{tikzcd} \star \arrow[r, bend left=50, "\mathbf{g}\,\mathbf{g}_1"] \arrow[r, bend right=50, "\mathbf{f}\,\mathbf{f}_1"'] \arrow[r, phantom, "\Uparrow h\,{}^{\mathbf{f}}h_1"] & \star \end{tikzcd}.$$

5. $\mathbf{i}_\star = 1_G$.

Let us check that the composition works: $\partial(h\,{}^{\mathbf{f}}h_1)\mathbf{f}\mathbf{f}_1 = \partial(h)\partial({}^{\mathbf{f}}h_1)\mathbf{f}\mathbf{f}_1 = \partial(h)\mathbf{f}\partial(h_1)\mathbf{f}^{-1}\mathbf{f}\mathbf{f}_1 = \mathbf{g}\mathbf{g}_1$.

We need the bicategory of Kapranov-Voevodsky finite-dimensional 2-vector spaces over a field $\mathbb{K}$ [11]. Let us describe 2-Vect$^{\mathbb{K}}$, a version of 2-vector spaces we find particularly useful:

1. The 0-objects are the natural numbers: $2\text{-Vect}^{\mathbb{K}}_0 = \mathbb{N}$ (we agree that $0 \in \mathbb{N}$).

2. The categories $2\text{-Vect}_1^{\mathbb{K}}(n,0)$ and $2\text{-Vect}_1^{\mathbb{K}}(0,m)$ are the trivial categories with one object.
3. For any two positive numbers $n$, $m$ the objects of the category $2\text{-Vect}_1^{\mathbb{K}}(n,m)$ are $n \times m$-matrices $(V_{i,j})$ of finite-dimensional $\mathbb{K}$-vector spaces.
4. The 2-morphisms in the category $2\text{-Vect}_1^{\mathbb{K}}(n,m)$ are

$$2\text{-Vect}_2^{\mathbb{K}}((V_{i,j}),(W_{i,j})) = \{ \begin{tikzcd} n \arrow[r, bend left=50, "W_{i,j}"] \arrow[r, bend right=50, "V_{i,j}"'] \arrow[r, phantom, "\Uparrow\varphi_{i,j}"] & m \end{tikzcd} \mid \varphi_{i,j} : V_{i,j} \to W_{i,j} \text{ is a linear map}\}$$

are $n \times m$-matrices $(\varphi_{i,j})$ of linear maps $\varphi_{i,j} : V_{i,j} \to W_{i,j}$. The composition of 2-morphisms is the composition of these linear maps:

$$(\varphi_{i,j}) \circ (\psi_{i,j}) = \begin{tikzcd} n \arrow[r, bend left=70, "W_{i,j}"] \arrow[r, "V_{i,j}" description] \arrow[r, bend right=70, "U_{i,j}"'] \arrow[r, phantom, shift left=4, "\Uparrow\varphi_{i,j}"] \arrow[r, phantom, shift right=4, "\Uparrow\psi_{i,j}"] & m \end{tikzcd} = \begin{tikzcd} n \arrow[r, bend left=50, "W_{i,j}"] \arrow[r, bend right=50, "U_{i,j}"'] \arrow[r, phantom, "\Uparrow\varphi_{i,j}\psi_{i,j}"] & m \end{tikzcd} .$$

5. The composition bifunctor $\diamondsuit_{n,p,m} : 2\text{-Vect}_1^{\mathbb{K}}(n,p) \times 2\text{-Vect}_1^{\mathbb{K}}(p,m) \to 2\text{-Vect}_1^{\mathbb{K}}(n,m)$ is

$$(\varphi_{i,j}) \diamond (\psi_{i,j}) = \begin{tikzcd} n \arrow[r, bend left=50, "W_{i,k}"] \arrow[r, bend right=50, "V_{i,k}"'] \arrow[r, phantom, "\Uparrow\varphi_{i,k}"] & p \arrow[r, bend left=50, "A_{k,j}"] \arrow[r, bend right=50, "B_{k,j}"'] \arrow[r, phantom, "\Uparrow\psi_{k,j}"] & m \end{tikzcd} = \begin{tikzcd}[column sep=huge] n \arrow[r, bend left=40, "\oplus_k W_{i,k}\otimes B_{k,j}"] \arrow[r, bend right=40, "\oplus_k V_{i,k}\otimes A_{k,j}"'] \arrow[r, phantom, "\Uparrow\oplus_k\varphi_{i,k}\otimes\psi_{k,j}"] & m \end{tikzcd} .$$

6. The associativity constraint is non-trivial: it arises from the associativity of tensor products of vector spaces.
7. $\mathbf{i}_n = (V_{i,j})$ where each $V_{i,i}$ is the field $\mathbb{K}$ and $V_{i,j} = 0$ if $i \neq j$.
8. The unitality constraints are non-trivial: they arise from the isomorphisms $\mathbb{K} \otimes V \cong V \cong V \otimes \mathbb{K}$.

There are other versions of this bicategory in the literature. There is a "more skeletal" version where one uses standard vector spaces $\mathbb{K}^n$ instead of all finite dimensional vector spaces as the matrix entries. We do not find this version very useful but a "bigger" version $\mathit{2}\text{-}\mathcal{V}ect^{\mathbb{K}}$ is convenient. In particular, $\mathit{2}\text{-}\mathcal{V}ect^{\mathbb{K}}$ is a 2-category. Let $\mathcal{A}^n$ be the category of finite dimensional representations of the semisimple commutative algebra $\mathbb{K}^n$. It is a semisimple $\mathbb{K}$-linear abelian category. The category $\mathcal{A}^1$ is the category of finite dimensional vector spaces. It has a monoidal structure via the tensor product of vector spaces. The category $\mathcal{A}^n$ is an $\mathcal{A}^1$-module category: the theory of module categories is developed by Ostrik [17]. Without repeating it here we spell out a few useful facts. The main feature of a module category $\mathcal{C}$ is an action bifunctor

$$\boxtimes : \mathcal{C} \times \mathcal{A}^1 \to \mathcal{C}.$$

Let $\mathcal{C} = A$-Mod for any associative $\mathbb{K}$-algebra $A$, not necessarily $\mathbb{K}^n$. The action bifunctor comes from the tensor product: if $M$ is an $A$-module, $V$ is a vector space, then $M \boxtimes V$ is an $A$-module, defined as a vector space $M \otimes_{\mathbb{K}} V$ with the $A$-action on the first factor.

A module functor from $\mathcal{C}_1$ to $\mathcal{C}_2$ is a functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ together with action intertwiners, certain functorial morphisms (i.e., given by a natural transformation of bifunctors)

$$F_{M,X} : F(M \boxtimes X) \longrightarrow F(M) \boxtimes X \quad \text{for all} \quad X \in \mathcal{A}^1, \ M \in \mathcal{C},$$

satisfying the standard natural conditions (pentagon and triangle commutativity) [17, Def 2.7]. A module functor $F$ is *strong* if all $F_{M,X}$ are isomorphisms. A module functor $F$ is *strict* if all $F_{M,X}$ are equalities.

A module natural transformation $\psi : F \Rightarrow G$ between two module functors $F, G : \mathcal{C}_1 \to \mathcal{C}_2$ is a natural transformation $\psi$ such that the following diagram is commutative for all $M \in \mathcal{C}_1$, $X \in \mathcal{A}^1$:

$$\begin{array}{ccc} F(M \boxtimes X) & \xrightarrow{F_{M,X}} & F(M) \boxtimes X \\ {\scriptstyle \psi_{M \boxtimes X}} \downarrow & & \downarrow {\scriptstyle \psi_M \boxtimes \mathrm{Id}_X} \\ G(M \boxtimes X) & \xrightarrow{G_{M,X}} & G(M) \boxtimes X. \end{array}$$

Let us give some examples of module functors and module natural transformations. If ${}_AP_B$ is a bimodule, the corresponding functor $F : B\text{-Mod} \to A\text{-Mod}$, $F(M) \coloneqq P \otimes_B M$ admits a canonical strong module functor structure:

$$F_{M,X} : F(M \boxtimes X) = P \otimes_B (M \otimes_{\mathbb{K}} V) \longrightarrow (P \otimes_B M) \otimes_{\mathbb{K}} V = F(M) \boxtimes X$$

is the standard associativity. A homomorphism of bimodules $\psi : {}_AP_B \to {}_AQ_B$ yields a module natural transformation

$$\psi : P \otimes_B \ \Rightarrow Q \otimes_B \ , \quad \psi_M(p \otimes_B m) = \psi(p) \otimes_B m \, .$$

Any additive functor (for instance, a Morita equivalence) $F : B\text{-Mod} \to A\text{-Mod}$ admits a non-canonical strong module functor structure. One chooses a basis in each vector space $X$ that gives isomorphisms $M \boxtimes X = M \otimes_{\mathbb{K}} X \cong \oplus M$ with a direct sum of copies of the module indexed by the basis. This leads to a strong module structure on $F$:

$$F_{M,X} : F(M \boxtimes X) \xrightarrow{\cong} F(\oplus M) \xrightarrow{\cong} \oplus F(M) \xrightarrow{\cong} F(M) \boxtimes X.$$

We are looking at the class of $\mathcal{A}^1$-module categories module-equivalent to $\mathcal{A}^n$. A module-equivalence admits a quasiinverse equivalence that is a strong module functor. This includes the zero category $\mathcal{A}^0$ that has only zero objects.

**Lemma 1.1.** *Let $\mathcal{C}$ be an $\mathcal{A}^1$-module category module-equivalent to $\mathcal{A}^n$. Let $L_1, \dots L_n$ be non-isomorphic simple objects in $\mathcal{C}$. Then for any object $M$ of $\mathcal{C}$ there is a functorial isomorphism*

$$M \cong \oplus_{i=1}^n L_i \boxtimes \mathcal{C}(L_i, M). \tag{$\clubsuit$}$$

*Proof.* Let $\psi : \mathcal{C} \to \mathcal{A}^n$ be a module equivalence, $\varphi$ its quasiinverse. Functorial isomorphism means that the identity functor and the functor in the right hand side of (♣) are naturally isomorphic. In $\mathcal{A}^n$ we have

$$\psi(M) \cong \oplus_{i=1}^n \psi(L_i) \boxtimes \mathcal{A}^n(\psi(L_i), \psi(M)) \cong \oplus_{i=1}^n \psi(L_i) \boxtimes \mathcal{C}(L_i, M).$$

The first isomorphism is given by the evaluation map. Now we apply $\varphi$, $\varphi_{M,X}$, and the quasiinverse data:

$$M \cong \varphi(\psi(M)) \cong \oplus_{i=1}^n \varphi(\psi(L_i)) \boxtimes \mathcal{C}(L_i, M) \cong \oplus_{i=1}^n L_i \boxtimes \mathcal{C}(L_i, M).$$

□

Lemma 1.1 replaces Ostrik's trick with internal hom-s [17]. Let us describe $\mathit{2}\text{-}\mathcal{V}ect^{\mathbb{K}}$ now. It is a subcategory of the 2-category of 2-modules $\mathit{2}\text{-}\mathcal{M}od^{\mathbb{K}}$.

1. The 0-objects $\mathit{2}\text{-}\mathcal{M}od_0^{\mathbb{K}}$ are $\mathcal{A}^1$-module categories module-equivalent to $A$-Mod for some associative $\mathbb{K}$-algebra $A$.
2. The objects of the category $\mathit{2}\text{-}\mathcal{M}od_1^{\mathbb{K}}(\mathcal{C}, \mathcal{D})$ are $\mathcal{A}^1$-module functors $\mathcal{C} \to \mathcal{D}$.
3. The 2-morphisms in the category $\mathit{2}\text{-}\mathcal{M}od_1^{\mathbb{K}}(\mathcal{C}, \mathcal{D})$ are $\mathcal{A}^1$-module natural transformations $F \Rightarrow G$.
4. The composition bifunctor $\mathit{2}\text{-}\mathcal{M}od_1^{\mathbb{K}}(\mathcal{C}, \mathcal{D}) \times \mathit{2}\text{-}\mathcal{M}od_1^{\mathbb{K}}(\mathcal{D}, \mathcal{E}) \to \mathit{2}\text{-}\mathcal{M}od_1^{\mathbb{K}}(\mathcal{C}, \mathcal{E})$ is the composition of functors: $F \diamond G := G \circ F = GF$. The action intertwiners are compositions too:

$$(F \diamond G)_{M,X} = G_{F(M),X} \circ G(F_{M,X}) : GF(M \boxtimes X) \to G(F(M) \boxtimes X) \to GF(M) \boxtimes X.$$

5. The associativity constraint is trivial. For instance, on the level of functorial morphisms

$$(FG)H_{M,X} = F_{GH(M),X} \circ F(G_{H(M),X}) \circ FG(H_{M,X}) = F(GH)_{M,X}.$$

In particular, $\mathit{2}\text{-}\mathcal{M}od^{\mathbb{K}}$ is a 2-category. The 2-category of 2-vector spaces $\mathit{2}\text{-}\mathcal{V}ect^{\mathbb{K}}$ is a full 2-subcategory of $\mathit{2}\text{-}\mathcal{M}od^{\mathbb{K}}$ whose 0-objects $\mathit{2}\text{-}\mathcal{V}ect_0^{\mathbb{K}}$ are those $\mathcal{A}^1$-module categories module-equivalent to $\mathcal{A}^n$ for some $n \in \mathbb{N}$. Now we are ready for the main theorem of this section.

**Theorem 1.2.** *The bicategories* 2-Vect$^{\mathbb{K}}$ *and* $\mathit{2}\text{-}\mathcal{V}ect^{\mathbb{K}}$ *are equivalent.*

*Proof.* Let us construct a 2-functor $F$ : 2-Vect$^{\mathbb{K}} \to \mathit{2}\text{-}\mathcal{V}ect^{\mathbb{K}}$. We choose non-isomorphic simple modules $L_1^{(n)}, \ldots L_n^{(n)}$ in $\mathcal{A}^n$ and set

$$F^0(n) = \mathcal{A}^n, \quad F^1_{n,m}((W_{i,j})_{n\times m}) : M \xrightarrow{♣} \bigoplus_i L_i^{(n)} \boxtimes \mathcal{A}^n(L_i^{(n)}, M) \mapsto \bigoplus_{i,j} L_j^{(m)} \boxtimes \mathcal{A}^n(L_i^{(n)}, M) \otimes W_{i,j}.$$

This defines functors $F^1_{n,m}$ on the objects. On the morphisms it acts on the coefficient vector spaces:

$$F^1_{n,m}((\varphi_{i,j}) : (V_{i,j}) \Rightarrow (W_{i,j})) = \bigoplus_{i,j} \mathrm{Id}_{L_j^{(m)}} \boxtimes \mathrm{Id}_{\mathcal{A}^n(L_i^{(n)},M)} \otimes \varphi_{i,j}.$$

The 2-isomorphism $F^2_n : \mathbf{i}_{\mathcal{A}^n} \Rightarrow F^1_{n,n}(\mathbf{i}_n)$ is (♣) combined with action isomorphisms $V \cong V \boxtimes \mathbb{K}$:

$$F^2_n(M) : M \xrightarrow{\clubsuit} \bigoplus_i L^{(n)}_i \boxtimes \mathcal{A}^n(L^{(n)}_i, M) \xrightarrow{\cong} \bigoplus_i L^{(n)}_i \boxtimes \mathcal{A}^n(L^{(n)}_i, M) \otimes \mathbb{K}$$

Finally, the compatibilities

$$F^2_{n,m,p} : \Diamond_{\mathcal{A}^n,\mathcal{A}^m,\mathcal{A}^p} \circ (F^1_{n,m} \times F^1_{m.p}) \Rightarrow F^1_{n,p} \circ \Diamond_{n,m,p}$$

boil down to associativities on the level of coefficient vector spaces:

$$F^2_{n,m,p}(U,V)(M) \ : \ \bigoplus_{i,j,k} L^{(p)}_k \boxtimes ( \Big( \mathcal{A}^m(L^{(m)}_j, (L^{(m)}_j \boxtimes \mathcal{A}^n(L^{(n)}_i, M) \otimes U_{i,j})) \Big) \otimes V_{j,k}) \xrightarrow{\cong} \qquad (\spadesuit)$$

$$\xrightarrow{\cong} \bigoplus_{i,j,k} L^{(p)}_k \boxtimes ( \Big( \mathcal{A}^n(L^{(n)}_i, M) \otimes U_{i,j} \Big) \otimes V_{j,k}) \xrightarrow{\cong} \bigoplus_{i,j,k} L^{(p)}_k \boxtimes (\mathcal{A}^n(L^{(n)}_i, M) \otimes (U_{i,j} \otimes V_{j,k})).$$

Let us now construct a quasiinverse 2-functor $G : \mathit{2}\text{-}\mathcal{V}ect^{\mathbb{K}} \to 2\text{-Vect}^{\mathbb{K}}$. In categories $\mathcal{C}$ and $\mathcal{D}$, $\mathcal{A}^1$-module equivalent to $\mathcal{A}^n$ and $\mathcal{A}^m$ correspondingly, we choose non-isomorphic simple objects $C_1, \dots C_n$ and $D_1, \dots D_m$. Let $H : \mathcal{C} \to \mathcal{D}$ be an $\mathcal{A}^1$-module functor. We use the functorial isomorphism (♣) to write down the 2-functor $G$:

$$G^0(\mathcal{C}) \coloneqq n, \quad G^0(\mathcal{D}) \coloneqq m, \quad G^1_{\mathcal{C},\mathcal{D}}(H) \coloneqq (V_{i,j}) \ \text{ where } \ V_{i,j} \coloneqq \mathcal{D}(D_j, H(C_i)).$$

On the 2-morphisms we reduce a natural transformation $\varphi : H \Rightarrow J$ to linear maps on coefficient spaces:

$$\bigoplus_j \mathrm{Id}_{D_j} \boxtimes \varphi_{i,j} : \bigoplus_j D_j \boxtimes \mathcal{D}(D_j, H(C_i)) \xrightarrow{\clubsuit} H(C_i) \xrightarrow{\varphi(C_i)} J(C_i) \xrightarrow{\clubsuit} \bigoplus_j D_j \boxtimes \mathcal{D}(D_j, J(C_i))$$

so that

$$G^1_{\mathcal{C},\mathcal{D}}(\varphi) \coloneqq (\varphi_{i,j}).$$

The 2-morphism $G^2_{\mathcal{C}} : \mathbf{i}_n \to G^1_{\mathcal{C},\mathcal{C}}(\mathbf{i}_{\mathcal{C}})$ is the obvious one. For $i \neq j$,

$$G^2_{\mathcal{C},i,i} \coloneqq \mathrm{Id} : \mathbb{K} \to \mathcal{C}(C_i, C_i) \cong \mathbb{K}, \quad G^2_{\mathcal{C},i,j} \coloneqq \mathrm{Id} : 0 \to \mathcal{C}(C_i, C_j) \cong 0.$$

Finally, let us define the compatibilities

$$G^2_{\mathcal{C},\mathcal{D},\mathcal{E}} : \Diamond_{n,m,p} \circ (G^1_{\mathcal{C},\mathcal{D}} \times G^1_{\mathcal{D},\mathcal{E}}) \Rightarrow G^1_{\mathcal{C},\mathcal{E}} \circ \Diamond_{\mathcal{C},\mathcal{D},\mathcal{E}}$$

for each pair of functors $H : \mathcal{C} \to \mathcal{D}$, $J : \mathcal{D} \to \mathcal{E}$. Let $E_1, \dots E_p$ be simple objects in $\mathcal{E}$. The linear maps

$$\varphi_{i,k} \coloneqq G^2_{\mathcal{C},\mathcal{D},\mathcal{E}}(H,J)_{i,k} : \bigoplus_j \mathcal{D}(D_j, H(C_i)) \otimes \mathcal{E}(E_k, J(D_j)) \to \mathcal{E}(E_k, JH(C_i))$$

come from the computation of compositions using the same trick as in (♠):

$$\bigoplus_k \mathrm{Id}_{E_k} \boxtimes \varphi_{i,k} : \bigoplus_{j,k} E_k \boxtimes \big( \mathcal{E}(E_k, J(D_j)) \otimes \mathcal{D}(D_j, H(C_i)) \big) \xrightarrow{\clubsuit} JH(C_i) \xrightarrow{\clubsuit} \bigoplus_k E_k \boxtimes \mathcal{E}(E_k, JH(C_i)).$$

It remains to write down the natural 2-isomorphisms $FG \Rightarrow \mathit{2}\text{-}\mathcal{V}ect^{\mathbb{K}}$ and $GF \Rightarrow 2\text{-Vect}^{\mathbb{K}}$. These 2-isomorphisms are straightforward, so we leave them as an exercise for an interested reader. □

We define *a 2-representation* of $\mathcal{K}$ as a 2-functor $R : \widetilde{\mathcal{K}} \to$ 2-Vect$^{\mathbb{K}}$ and *a 2-module* for $\mathcal{K}$ as a 2-functor $\mathcal{R} : \widetilde{\mathcal{K}} \to \mathit{2}\text{-}\mathcal{M}od^{\mathbb{K}}$. Since 2-Vect$^{\mathbb{K}}$ and $\mathit{2}\text{-}\mathcal{V}ect^{\mathbb{K}}$ are equivalent, we can think of a 2-representation as a "semisimple" 2-module $\mathcal{R} : \widetilde{\mathcal{K}} \to \mathit{2}\text{-}\mathcal{V}ect^{\mathbb{K}}$. The latter approach is convenient for construction of 2-representations. The former approach is convenient for analysis of 2-representations. Let us summarise what information a 2-representation (the former version) contains [16, Definition 1].

1. A number $n = R^0(\star)$. We call this number *the degree* of $R$:
2. A 1-morphism $R^1(\mathbf{g}) = R^1_{\star,\star}(\mathbf{g}) = (V_{i,j}) : n \to n$ for every $\mathbf{g} \in G$. The dimensions of these vector spaces form a matrix $\dim(R^1(\mathbf{g})) \in \mathbb{N}^{n\times n}$.
3. 2-Isomorphisms $R^1(\mathbf{g}, x) = R^1_{\star,\star}(\mathbf{g} \xRightarrow{x} \partial(x)\mathbf{g}) = (\varphi_{i,j}) : R^1(\mathbf{g}) \Rightarrow R^1(\partial(x)\mathbf{g})$ for all $x \in H$, $\mathbf{g} \in G$ that are subject to vertical multiplicativity rule $R^1(\partial(x)\mathbf{g}, y)R^1(\mathbf{g}, x) = R^1(\mathbf{g}, yx)$.
4. A 2-isomorphism $R^2_\star = (\theta_{i,j}) : \mathbf{i}_n \Rightarrow R(1_G)$.
5. 2-Isomorphisms $R^2(\mathbf{f}, \mathbf{g}) = R^2_{\star,\star,\star}(\mathbf{f}, \mathbf{g}) = (\psi_{i,j}) : R^1(\mathbf{f}) \diamond R^1(\mathbf{g}) \Rightarrow R^1(\mathbf{fg})$ for every pair $\mathbf{f}, \mathbf{g} \in G$ such that the pentagon (or rather degenerated due to strictness of $\widetilde{\mathcal{K}}$ hexagon) diagram with two possible 2-morphisms $(R^1(\mathbf{f}) \diamond R^1(\mathbf{g})) \diamond R^1(\mathbf{h}) \Rightarrow R^1(\mathbf{fgh})$ and the triangle diagrams with two possible 2-morphisms $R^1(\mathbf{f}) \diamond R^1(1_G) \Rightarrow R^1(\mathbf{f})$ and $R^1(1_G) \diamond R^1(\mathbf{g}) \Rightarrow R^1(\mathbf{g})$ are all commutative. As this is a natural transformation of bifunctors, the naturality condition

$$R^2(\partial(x)\mathbf{f}, \partial(y)\mathbf{g}) \circ \left(R^1(\mathbf{f}, x) \diamond R^1(\mathbf{g}, y)\right) = R^1(\mathbf{fg}, x\,{}^{\mathbf{f}}y) \circ R^2(\mathbf{f}, \mathbf{g}) \ ,$$

ought to be observed.

We call the 2-representation $R$ *unital* if $R^2_\star$ is an identity and *strict* if all $R^2(\mathbf{f}, \mathbf{g})$ and $R^2_\star$ are identities. We apply the same adjectives to a 2-module under the similar conditions.

Item 5 ensures that $R^1(\mathbf{g}) \diamond R^1(\mathbf{f})$ is 2-isomorphic to $R^1(\mathbf{gf})$ which implies that $\dim \circ R^1$ is a group homomorphism from $G$ to $\mathrm{GL}_n(\mathbb{N}) \cong S_n$. Item 3 further necessitates $\dim(R^1(\mathbf{g})) = \dim(R^1(\partial(x)\mathbf{g}))$. Hence, hidden inside a 2-representation of $\mathcal{K}$ we find a permutation action of the fundamental group $\pi_1(\mathcal{K}) = G/\partial(H)$ on the finite set $\{1, 2, \dots n\}$. It is fruitful to think of a 2-representation of $\mathcal{K}$ as a permutation action of $\pi_1(\mathcal{K})$ together with some additional data. The precise nature of this data will be uncovered later (cf. Section 3).

A homomorphism of 2-representations $\psi : R \to R'$ or 2-modules $\psi : \mathcal{R} \to \mathcal{R}'$ is a natural 2-transformation of 2-functors. An equivalence of 2-representations is a natural 2-isomorphism of 2-functors. Let 2-Rep$^n(\mathcal{K})$ be the class of equivalence classes of 2-representations of $\widetilde{\mathcal{K}}$ of degree $n$. It is clear from the description above that 2-Rep$^n(\mathcal{K})$ is actually a set.

Before we proceed with our studies, we remark that the 2-representations of $\mathcal{K}$ can be considered over any semisimple rigid monoidal category $\mathcal{C}$. Instead of 2-Vect$^{\mathbb{K}}$ one considers square matrices of objects of $\mathcal{C}$. Instead of $\mathit{2}\text{-}\mathcal{V}ect^{\mathbb{K}}$ one considers semisimple $\mathcal{C}$-module categories with finitely many simple objects under their Ostrik's internal hom-s [17]. The results of this section easily extend to this greater generality.

## 2. Morita theory for a 2-group

An associative algebra $A$ admits a crossed module of 2-automorphisms $2\text{-Aut}(A) \coloneqq (A^\times \xrightarrow{\partial} \mathrm{Aut}(A))$ where $A^\times$ is the group of units of $A$ and $\partial(x)$ is the inner automorphism $y \mapsto xyx^{-1}$. Let $\mathcal{K} = (H \xrightarrow{\partial} G)$ be a crossed module. By *a $\mathcal{K}$-algebra* we understand an associative algebra $A$ with a (left) action of $\mathcal{K}$, i.e., a crossed module homomorphism $\omega_A : \mathcal{K} \to 2\text{-Aut}(A)$.

It is useful to introduce a weak version of this concept: *a weak $\mathcal{K}$-algebra* is an associative algebra $A$ with a weak crossed module homomorphism $\omega_A : \mathcal{K} \to 2\text{-Aut}(A)$. We define *a weak crossed module homomorphism* $\omega_A : \mathcal{K} \to 2\text{-Aut}(A)$ as a triple $(\omega_1, \omega_2, \omega_3)$ where $\omega_1 : G \to \mathrm{Aut}(A)$ is a group homomorphism, $\omega_3 : G \times G \to Z(A)^\times$ is a normalised cocycle, i.e.,

$$\omega_3(\mathbf{g}, 1) = \omega_3(1, \mathbf{g}) = 1, \ \ {}^{\mathbf{f}}\omega_3(\mathbf{g}, \mathbf{h})\omega_3(\mathbf{f}, \mathbf{gh}) = \omega_3(\mathbf{fg}, \mathbf{h})\omega_3(\mathbf{f}, \mathbf{g}) \text{ for all } \mathbf{f}, \mathbf{g}, \mathbf{h} \in G,$$

and $\omega_2 : H \to A^\times$ is a unital projective group homomorphism with the cocycle $\omega_3 \circ (\partial \times \partial)$, i.e.,

$$\omega_2(xy) = \omega_3(\partial x, \partial y)\omega_2(x)\omega_2(y) \text{ for all } x, y \in H$$

such that they respect the crossed module structure maps:

$$\omega_1(\partial x) = \partial(\omega_2(x)), \ \ \omega_2({}^{\mathbf{f}}x) = {}^{\omega_1(\mathbf{f})}\omega_3(\partial x, \mathbf{f}^{-1})\omega_3(\mathbf{f}, \partial x\, \mathbf{f}^{-1})\omega_3(\mathbf{f}, \mathbf{f}^{-1})^{-1}\, {}^{\omega_1(\mathbf{f})}\omega_2(x)\ .$$

Note that the normalised cocycle condition implies $\omega_2(1) = 1$. Also observe that if $H$ is trivial (so that $\widetilde{\mathcal{K}}$ is just the group $G$), a weak homomorphism is just a homomorphism of $G$ with a normalised cocycle.

Weak crossed module homomorphisms have been studied by Noohi [15, Def. 8.4] but our concept is different. Noohi's weak homomorphism is also a triple $(\omega_1, \omega_2, \omega_3)$ of the maps with the same domains and ranges. The difference is that $\omega_2$ is now required to be a homomorphism of groups, while $\omega_1$ is a projective homomorphism with the cocycle $\partial \circ \omega_3$. There is a different respect condition. Since both notions yield homomorphisms of 2-groups, it is plausible that there may be a common generalisation. A promising concept is a quadruple $(\omega_1, \omega_2, \omega_3, \omega_3')$ where $\omega_2$ is a projective homomorphism with the cocycle $\omega_3 \circ (\partial \times \partial)$ and $\omega_1$ is a projective homomorphism with the cocycle $\partial \circ \omega_3'$. These quadruples need to define homomorphisms of the corresponding 2-groups. We leave it at that hoping that future research will clarify a connection between these two notions of weak homomorphism.

Let $A$ be a weak $\mathcal{K}$-algebra. The group $G$ acts on the category of left $A$-modules $A$-Mod on the right [9]. Each automorphism $\mathbf{g} \in \mathrm{Aut}(A)$ gives an endofunctor $[\mathbf{g}] : A\text{-Mod} \to A\text{-Mod}$. For an $A$-module $M$, the twisted $A$-module $M^{[\mathbf{g}]}$ is equal to $M$ as a vector space and the new action: $a \cdot^{[\mathbf{g}]} m = \mathbf{g}(a)m$. On morphisms $\varphi^{[\mathbf{g}]} = \varphi$. Notice that $[\mathbf{f}] \circ [\mathbf{g}] = [\mathbf{gf}]$, so that this is a right action.

We would like to extend this to an action of the 2-group $\widetilde{\mathcal{K}}$. With this in mind, we consider the 2-group 2-$\mathcal{A}ut$($A$-Mod) of the automorphisms of the category of left $A$-modules:

1. The 0-objects of 2-$\mathcal{A}ut$($A$-Mod) is a one point set $\{\star\}$.

2. The 1-objects $2\text{-}\mathcal{A}ut(A\text{-Mod})_1(\star,\star)$ are strong $\mathcal{A}_1$-module autoequivalences of $A$-Mod.

3. For a pair of autoequivalences $F_1, F_2$ the 2-morphisms $2\text{-}\mathcal{A}ut(A\text{-Mod})_2(F_1, F_2)$ are $\mathcal{A}_1$-module natural isomorphisms $F_1 \Rightarrow F_2$.

4. The composition bifunctor is the composition of functors $F_1 \diamond F_2 := F_2 \circ F_1 = F_2 F_1$.

This 2-group is not strict because functors have quasi-inverses, not inverses, in general. Another way to think of $2\text{-}\mathcal{A}ut(A\text{-Mod})$ is a 2-subcategory of $\mathit{2}\text{-}\mathcal{M}od^{\mathbb{K}}$ with one 0-object.

Before we proceed, let us describe the natural transformations between the group twists.

**Lemma 2.1.** *Suppose* $\mathbf{f}$ *and* $\mathbf{g}$ *are automorphisms of a ring* $A$. *Then the map*

$$\Upsilon : \{x \in A \mid \forall a \in A \;\; xa = \mathbf{g}(\mathbf{f}^{-1}(a))x\} \to \text{Nat.Trans}([\mathbf{f}], [\mathbf{g}]), \quad \Upsilon(x)_M : m \mapsto x \cdot m$$

*is a bijection.*

*Proof.* Observe that $\Upsilon(x)_M \in \hom(M^{[\mathbf{f}]}, M^{[\mathbf{g}]})$ for any $A$-module $M$:

$$\Upsilon(x)_M(a \cdot^{[\mathbf{f}]} m) = x\mathbf{f}(a)m = \mathbf{g}(\mathbf{f}^{-1}(\mathbf{f}(a)))xm = \mathbf{g}(a)xm = a \cdot^{[\mathbf{g}]} \Upsilon(x)_M(m)$$

for all $m \in M$, $a \in A$. Since $F(xm) = xF(m)$ for any homomorphism $F$, $\Upsilon(x)$ is a natural transformation. Hence, $\Upsilon$ is a well-defined function.

To show that this is a bijection we construct the inverse function:

$$\Xi : \text{Nat.Trans}([\mathbf{f}], [\mathbf{g}]) \to \{x \in A \mid \forall a \in A \;\; xa = \mathbf{g}(\mathbf{f}^{-1}(a))x\}, \quad \Xi(\varphi) = \varphi_A(1).$$

The equality $\Xi\Upsilon = \text{Id}$ is obvious: $\Xi(\Upsilon(x)) = \Upsilon(x)_A(1) = x$. The opposite equality $\Upsilon\Xi = \text{Id}$ follows from the fact that ${}_AA$ is a generator of $A$-Mod so that a natural transformation $\varphi$ is uniquely determined by $\varphi_A$: using the $A$-module homomorphism $F : A \to M$, $F(a) = am$, we conclude that $\varphi_M(m) = \varphi_M(F(1)) = F(\varphi_A(1)) = \varphi_A(1)m$. $\square$

Armed with an understanding of natural transformations between the twists by automorphisms, we can mould $A$-Mod into a 2-module for $\widetilde{2\text{-Aut}}(A)$:

**Proposition 2.2.** *The assignment* $\Theta(\mathbf{g}) = [\mathbf{g}]$ *extends to a strict 2-module* $\Theta : \widetilde{2\text{-Aut}}(A) \to 2\text{-}\mathcal{A}ut(A\text{-Mod})$.

*Proof.* Inevitably, $\Theta^0(\star) = \star$. We define the functor $\Theta^1_{\star,\star}$ using the map from Lemma 2.1:

$$\Theta^1_{\star,\star}(\mathbf{g}) := [\mathbf{g}], \quad \Theta^1_{\star,\star}(\mathbf{g}_1 \overset{x}{\Longrightarrow} \mathbf{g}_2) = \Theta^1_{\star,\star}\Bigg( \begin{tikzcd} \star \arrow[r, bend left=50, "\mathbf{g}_2"{name=U}] \arrow[r, bend right=50, "\mathbf{g}_1"'{name=D}] & \star \arrow[from=D, to=U, Rightarrow, "x"] \end{tikzcd} \Bigg) := \Upsilon(x).$$

Since each $[\mathbf{g}]$ is an $\mathcal{A}^1$-module functor, $\Theta^1_{\star,\star}$ is well-defined. Observe that $\mathbf{g}_2 = \partial(x)\mathbf{g}_1$ so that $xa = xax^{-1}x = \mathbf{g}_2(\mathbf{g}_1^{-1}(a))x$ for all $a \in A$ and $\Upsilon(x)$ is a well-defined $\mathcal{A}^1$-module natural transformation.

Let us verify that $\Theta^1_{\star,\star}$ is functor:

$$\Theta^1_{\star,\star}([\mathbf{g}_2 \xRightarrow{y} \mathbf{g}_3] \circ [\mathbf{g}_1 \xRightarrow{x} \mathbf{g}_2]) = \Theta^1_{\star,\star}(\mathbf{g}_1 \xRightarrow{yx} \mathbf{g}_3) = \Upsilon(yx) = \Upsilon(y)\Upsilon(x) = \Theta^1_{\star,\star}(\mathbf{g}_2 \xRightarrow{y} \mathbf{g}_3)\Theta^1_{\star,\star}(\mathbf{g}_1 \xRightarrow{x} \mathbf{g}_2)\ .$$

The 2-isomorphism $\Theta^2_\star$ and the compatibilities $\Theta^2_{\star,\star,\star}$ can be chosen to be identities making $\Theta$ strict. $\square$

Now we are ready "to compose" the canonical homomorphism of 2-groups $\Theta : \widetilde{2\text{-Aut}}(A) \to 2\text{-}\mathcal{A}ut(A\text{-Mod})$ with a weak crossed module homomorphism $\omega_A : \mathcal{K} \to 2\text{-Aut}(A)$. We have to deal repeatedly with expressions of the form $[\mathbf{f}] \diamond \Upsilon(x)$ that we transform to $\Upsilon({}^{\mathbf{f}}x)$. It is instructive to apply both expression to an element $m \in M$. The former gives $x \cdot^{[\mathbf{f}]} m$ while the latter gives $({}^{\mathbf{f}}x) \cdot m$. Hence, they are equal.

**Proposition 2.3.** *The structure of a weak $\mathcal{K}$-algebra on $A$ gives rise to a unital 2-module $\Theta_A : \widetilde{\mathcal{K}} \to 2\text{-}\mathcal{A}ut(A\text{-Mod})$ defined as follows:*

$$\Theta^0_A(\star) = A\text{-Mod}, \quad \Theta^1_{A\ \star,\star}(\mathbf{g}) = [\omega_1(\mathbf{g})], \quad \Theta^1_A(\mathbf{g}, x) = \Theta^1_{A\ \star,\star}(\mathbf{g} \xRightarrow{x} \partial(x)\mathbf{g}) = \Upsilon(\omega_3(\partial x, \mathbf{g})\omega_2(x)),$$

$$\Theta^2_{A\ \star} = Id, \quad \Theta^2_A(\mathbf{f}, \mathbf{g}) = \Theta^2_{A\ \star,\star,\star}(\mathbf{f}, \mathbf{g}) = \Upsilon(\omega_3(\mathbf{f}, \mathbf{g})).$$

*If $A$ is a $\mathcal{K}$-algebra, $\Theta_A$ is a strict 2-module.*

*Proof.* Lemma 2.1 guarantees that the 2-isomorphisms are well-defined: $\Theta^1_A(\mathbf{g}, x)$ acts $\Theta^1_A(\mathbf{g}) \Rightarrow \Theta^1_A(\partial(x)\mathbf{g})$, while $\Theta^2_A(\mathbf{f}, \mathbf{g})$ acts $\Theta^1_A(\mathbf{f}) \diamond \Theta^1_A(\mathbf{g}) \Rightarrow \Theta^1_A(\mathbf{fg})$. The vertical multiplicativity follows from the definition of a weak homomorphism of crossed modules:

$$\Theta^1_A(\partial x\, \mathbf{g}, y)\Theta^1_A(\mathbf{g}, x) = \Upsilon(\omega_3(\partial y, \partial x\, \mathbf{g})\omega_2(y))\Upsilon(\omega_3(\partial x, \mathbf{g})\omega_2(x)) = \Upsilon(\omega_3(\partial y, \partial x\, \mathbf{g})\, {}^{\partial y}\omega_3(\partial x, \mathbf{g})\omega_2(y)\omega_2(x))$$

$$= \Upsilon(\omega_3(\partial y\, \partial x, \mathbf{g})\omega_3(\partial y, \partial x)\omega_2(y)\omega_2(x)) = \Upsilon(\omega_3(\partial(yx), \mathbf{g})\omega_2(yx)) = \Theta^1_A(\mathbf{g}, yx).$$

The equality between the lines is the cocycle condition. A similar reasoning proves the naturality condition for $\Theta^2_{A\ \star,\star,\star}$. Checking it, we use the centrality property ${}^{\partial x}\omega_3(\mathbf{f}, \mathbf{g}) = \omega_3(\mathbf{f}, \mathbf{g})$. To help the reader we underline terms undergoing transformations at the next equality.

$$\Theta^2_A(\partial(x)\mathbf{f}, \partial(y)\mathbf{g}) \circ \big(\Theta^1_A(\mathbf{f}, x) \diamond \Theta^1_A(\mathbf{g}, y)\big) = \Upsilon(\omega_3(\partial x\, \mathbf{f}, \partial y\, \mathbf{g}))\Upsilon(\underline{{}^{\partial x\, \mathbf{f}}[\omega_3(\partial y, \mathbf{g})\omega_2(y)]})\cdot$$

$$\cdot\Upsilon(\omega_3(\partial x, \mathbf{f})\omega_2(x)) = \Upsilon\Big(\underline{\omega_3(\partial x\, \mathbf{f}, \partial y\, \mathbf{g})\omega_3(\partial x, \mathbf{f})}\, {}^{\mathbf{f}}\omega_3(\partial y, \mathbf{g})\, \underline{{}^{\partial x\, \mathbf{f}}\omega_2(y)\omega_2(x)}$$

$$= \omega_3(\partial x, \mathbf{f}\partial y\, \mathbf{g})\underline{\omega_3(\mathbf{f}, \partial y\, \mathbf{g})\, {}^{\mathbf{f}}\omega_3(\partial y, \mathbf{g})}\omega_2(x)\, \underline{{}^{\mathbf{f}}\omega_2(y)} = \omega_3(\partial x, \mathbf{f}\partial y\, \mathbf{g})\underline{\omega_3(\mathbf{f}, \partial y)}\omega_3(\mathbf{f}\partial y, \mathbf{g})\omega_2(x)\cdot$$

$$\cdot\underline{{}^{\mathbf{f}}\omega_3(\partial y, \mathbf{f}^{-1})^{-1}\omega_3(\mathbf{f}, \partial y\, \mathbf{f}^{-1})^{-1}}\ \underline{\omega_3(\mathbf{f}, \mathbf{f}^{-1})}\omega_2({}^{\mathbf{f}}y) = \omega_3(\partial x, \mathbf{f}\partial y\, \mathbf{g})\underline{\omega_3(\mathbf{f}\partial y, \mathbf{f}^{-1})^{-1}\, {}^{\mathbf{f}}\omega_3(\mathbf{f}^{-1}, \mathbf{f})}$$

$$\omega_3(\mathbf{f}\partial y, \mathbf{g})\omega_2(x)\omega_2({}^{\mathbf{f}}y) = \omega_3(\partial x, \mathbf{f}\partial y\, \mathbf{g})\underline{\omega_3(\mathbf{f}\partial y\, \mathbf{f}^{-1}, \mathbf{f})\omega_3(\mathbf{f}\partial y\, \mathbf{g})}\omega_2(x)\omega_2({}^{\mathbf{f}}y) =$$

$$\underline{\omega_3(\partial x, \mathbf{f}\partial y\, \mathbf{g})\omega_3(\mathbf{f}\partial y\, \mathbf{f}^{-1}, \mathbf{fg})}\omega_3(\mathbf{f}, \mathbf{g})\omega_2(x)\omega_2({}^{\mathbf{f}}y) = \underline{\omega_3(\partial x\, \mathbf{f}\partial y\, \mathbf{f}^{-1}, \mathbf{fg})}\omega_3(\mathbf{f}, \mathbf{g})\underline{\omega_3(\partial x, \mathbf{f}\partial y\, \mathbf{f}^{-1})\omega_2(x)\omega_2({}^{\mathbf{f}}y)}\Big)$$

$$= \Upsilon(\omega_3(\partial(x\, {}^{\mathbf{f}}y), \mathbf{fg})\omega_2(x\, {}^{\mathbf{f}}y))\Upsilon(\omega_3(\mathbf{f}, \mathbf{g})) = \Theta^1_A(\mathbf{fg}, x\, {}^{\mathbf{f}}y) \circ \Theta^2_A(\mathbf{f}, \mathbf{g})\ .$$

Let us now verify commutativity of the pentagon diagram. One of the key 2-morphisms $(\Theta^1_A(\mathbf{f}) \diamond \Theta^1_A(\mathbf{g})) \diamond \Theta^1_A(\mathbf{h}) \Rightarrow \Theta^1_A(\mathbf{fgh})$ is

$$\Upsilon(\omega_3(\mathbf{fg},\mathbf{h}))\Upsilon(\omega_3(\mathbf{f},\mathbf{g})) = \Upsilon(\omega_3(\mathbf{fg},\mathbf{h})\omega_3(\mathbf{f},\mathbf{g})).$$

The second key 2-morphism is

$$\Upsilon(\omega_3(\mathbf{f},\mathbf{gh})) \circ \Big([\mathbf{f}] \diamond \Upsilon(\omega_3(\mathbf{g},\mathbf{h}))\Big) = \Upsilon(\omega_3(\mathbf{f},\mathbf{gh})\, {}^{\mathbf{f}}\omega_3(\mathbf{g},\mathbf{h})).$$

They are equal by the cocycle property of $\omega_3$. Commutativity of the triangle diagrams is obvious because $\Theta^2_{A\ \star} = \mathrm{Id}$. This finishes the proof of the first statement.

A $\mathcal{K}$-algebra is a weak $\mathcal{K}$-algebra with trivial $\omega_3$. Thus, all $\Theta^2_A(\mathbf{f},\mathbf{g})$ are trivial and the homomorphism $\Theta_A$ is strict. This proves the second statement. □

The following *Extension Lemma* is a partially converse statement to Proposition 2.3.

**Lemma 2.4.** *Let $\mathcal{K} = (H \to G)$ be a crossed module, $A$ an associative $G$-algebra. Suppose we have a unital 2-module $\Omega : \widetilde{\mathcal{K}} \to 2\text{-}\mathcal{A}ut(A\text{-Mod})$ whose restriction to $G$ comes from an action $\varpi : G \to \mathrm{Aut}(A)$:*

$$\Omega^1_{\star,\star}(\mathbf{g}) = [\varpi(\mathbf{g})] \quad \textit{for all } \mathbf{g} \in G.$$

*Then there exists a weak $\mathcal{K}$-algebra structure on $A$ such that $\Omega = \Theta_A$ (the latter is defined in Proposition 2.3). Furthermore, if $\Omega$ is strict, then $A$ is a $\mathcal{K}$-algebra.*

*Proof.* The first component of a weak homomorphism $(\omega_1, \omega_2, \omega_3) : \widetilde{\mathcal{K}} \to 2\text{-}\mathrm{Aut}(A)$ is already defined: $\omega_1(\mathbf{g}) \coloneqq \varpi(\mathbf{g})$. Let us define the other two components using Lemma 2.1:

$$\omega_2(x) \coloneqq \Upsilon^{-1}\Big(\Omega^1_{\star,\star}(1_G \xrightarrow{x} \partial(x))\Big), \quad \omega_3(\mathbf{f},\mathbf{g}) \coloneqq \Upsilon^{-1}\Big(\Omega^2_{\star,\star,\star}(\mathbf{f},\mathbf{g})\Big)$$

for all $\mathbf{f},\mathbf{g} \in G$, $x \in H$. Let us observe that $\Omega$ can be computed from the triple $(\omega_1,\omega_2,\omega_3)$ by formulas for $\Theta_A$ in Proposition 2.3. The only formula that requires checking is the following:

$$\Omega^1_{\star,\star}(\mathbf{g},x) = \Omega^1_{\star,\star}([1 \xrightarrow{x} \partial(x)] \diamond [\mathbf{g} \xrightarrow{1} \mathbf{g}]) = \Omega^2_{\star,\star,\star}(\partial x, \mathbf{g}) \circ (\Omega^1_{\star,\star}(1,x) \diamond \Omega^1_{\star,\star}(\mathbf{g},1)) = \Upsilon(\omega_3(\partial x,\mathbf{g})\omega_2(x)).$$

We have used the fact that $\Omega^1_{\star,\star}(\mathbf{g},1)$ is identity, that follows from vertical multiplicativity.

It remains to verify that $(\omega_1,\omega_2,\omega_3)$ is a weak homomorphism of crossed modules. This boils down to checking the following five identities:

1. $\omega_1(\partial x) = \partial(\omega_2(x))$: Let $a = \omega_2(x)$. Pick $b \in A$. Observe that $\Omega^1_{\star,\star}(1_G \xrightarrow{x} \partial(x))_A(b) = ab$. Hence,

$$ab = \Omega^1_{\star,\star}(1_G \xrightarrow{x} \partial(x))_A(b \cdot 1) = b \cdot^{[\partial x]} \Omega^1_{\star,\star}(1_G \xrightarrow{x} \partial(x))_A(1) = {}^{\partial x}b \cdot a\ .$$

We are done since $a$ is inevitably invertible.

2. $\omega_2({}^{\mathbf{f}}x) = \ \dots \ {}^{\omega_1(\mathbf{f})}\omega_2(x)$: This follows from the definition of a 2-functor:

$$\omega_2({}^{\mathbf{f}}x) = \Omega^1_{\star,\star}(1 \xRightarrow{{}^{\mathbf{f}}x} \mathbf{f}\partial x\,\mathbf{f}^{-1}) = \Omega^1_{\star,\star}\Bigg( \begin{tikzcd} \star \arrow[r, bend left=50, "\mathbf{f}"] \arrow[r, bend right=50, "\mathbf{f}"'] \arrow[r, phantom, "1\Uparrow"] & \star \arrow[r, bend left=50, "\partial x\,\mathbf{f}^{-1}"] \arrow[r, bend right=50, "\mathbf{f}^{-1}"'] \arrow[r, phantom, "x\Uparrow"] & \star \end{tikzcd} \Bigg) =$$

$$= \Omega^2_{\star,\star,\star}(\mathbf{f}, \partial x\,\mathbf{f}^{-1}) \circ (\Omega^1_{\star,\star}(\mathbf{f}, 1) \diamond \Omega^1_{\star,\star}(\mathbf{f}^{-1}, x)) \circ \Omega^2_{\star,\star,\star}(\mathbf{f}, \mathbf{f}^{-1})^{-1} = \omega_3(\mathbf{f}, \partial x\,\mathbf{f}^{-1})\cdot$$

$$\cdot\, {}^{\mathbf{f}}(\omega_3(\partial x, \mathbf{f}^{-1})\omega_2(x))\omega_3(\mathbf{f}, \mathbf{f}^{-1})^{-1} = \omega_3(\mathbf{f}, \partial x\,\mathbf{f}^{-1})\, {}^{\mathbf{f}}\omega_3(\partial x, \mathbf{f}^{-1})\omega_3(\mathbf{f}, \mathbf{f}^{-1})^{-1}\, {}^{\mathbf{f}}\omega_2(x).$$

3. $\omega_2(xy) = \omega_3(\partial x, \partial y)\omega_2(x)\omega_2(y)$: Again, this follows from the definition of a 2-functor:

$$\omega_2(xy) = \Omega^1\Bigg( \begin{tikzcd} \star \arrow[r, bend left=50, "\partial x"] \arrow[r, bend right=50, "1"'] \arrow[r, phantom, "x\Uparrow"] & \star \arrow[r, bend left=50, "\partial y"] \arrow[r, bend right=50, "1"'] \arrow[r, phantom, "y\Uparrow"] & \star \end{tikzcd} \Bigg) = \Omega^2_{\star,\star,\star}(\partial x, \partial y) \circ (\Omega^1_{\star,\star}(1, x) \diamond \Omega^1_{\star,\star}(1, y))$$

$$= \omega_3(\partial x, \partial y)\, {}^{\partial x}\omega_2(y)\omega_2(x). = \omega_3(\partial x, \partial y)\omega_2(x)\omega_2(y)\omega_2(x)^{-1}\omega_2(y) = \omega_3(\partial x, \partial y)\omega_2(x)\omega_2(y).$$

4. ${}^{\mathbf{f}}\omega_3(\mathbf{g}, \mathbf{h})\omega_3(\mathbf{f}, \mathbf{gh}) = \omega_3(\mathbf{fg}, \mathbf{h})\omega_3(\mathbf{f}, \mathbf{g})$ : This follows from commutativity of the pentagon (collapsed hexagon) diagram. The argument is the converse of the argument given in the proof of Proposition 2.3.
5. $\omega_3(\mathbf{g}, 1) = 1 = \omega_3(1, \mathbf{g})$: Both identities follow from the 2-module being unital.

The second statement is immediate. If $\Omega^2_{\star,\star,\star}$ are all trivial, then $\omega_3 \equiv 1$. Thus, $(\omega_1, \omega_2, 1)$ is a homomorphism of crossed modules. □

It is a moot point whether it is possible to replace weak $\mathcal{K}$-algebras with $\mathcal{K}^{\maltese}$-algebras for a new "centrally extended" crossed module $\mathcal{K}^{\maltese} = (H^{\maltese} \to G)$ where $1 \to C \to H^{\maltese} \to H \to 1$ is a $G$-equivariant central extension. It is certainly possible to do it for a single weak $\mathcal{K}$-algebra $A$: one can use $C = Z(A)^{\times}$ to achieve this goal. However, if one wishes a single $H^{\maltese}$ to serve all weak $\mathcal{K}$-algebras, this becomes subtly dependent on $\mathcal{K}$. To achieve this $H^{\maltese}$ needs to be a $G$-equivariant stem cover of $H$. Such thing may exist, say if $H$ is perfect. However, it does not exist, in general: Derek Holt has shown us an example of finite $H$ and $G$ where such cover does not exist.

The unitality assumption is not onerous: we can always make a 2-module unital as explained in the following lemma, whose proof is routine. As the referee has pointed out, the second statement is a special case of the general fact that any monoidal functor between monoidal categories is monoidally equivalent to a unital monoidal functor.

**Lemma 2.5.** *If* $R : \widetilde{\mathcal{K}} \to 2\text{-}\mathcal{A}ut(A\text{-Mod})$ *is a 2-module with* $R^1_{\star,\star}(\mathbf{g}) = [\varpi(\mathbf{g})]$ *for some group homomorphism* $\varpi : G \to \mathrm{Aut}(A)$*, then*

$$R^2(\mathbf{f}, 1) = R^2(1, \mathbf{g}) = R^2(1, 1) = \Upsilon(z)$$

*for some element $z$ of the centre $Z(A)$. Moreover, $R$ is equivalent to a unital 2-module $\Theta : \widetilde{\mathcal{K}} \to 2\text{-}\mathcal{A}ut(A\text{-Mod})$ defined by*

$$\Theta^1 = R^1, \quad \Theta^2_\star = R^2_\star, \quad \Theta^2(\mathbf{f}, \mathbf{g}) = \Upsilon(z^{-1}) \circ R^2(\mathbf{f}, \mathbf{g}) \ .$$

For the rest of the section we concentrate on $\mathcal{K}$-algebras. Our goal is Morita Theory that will pinpoint when two $\mathcal{K}$-algebras give equivalent 2-modules.

We consider a Morita equivalence $\Phi : A\text{-Mod} \to B\text{-Mod}$ together with its quasiinverse $\Phi^{-1} : B\text{-Mod} \to A\text{-Mod}$. We suppose both are equipped with strong module structures. This data defines a homomorphism of 2-groups

$$\widetilde{\Phi} : 2\text{-}\mathcal{A}ut(A\text{-Mod}) \to 2\text{-}\mathcal{A}ut(B\text{-Mod}), \quad \widetilde{\Phi}^0(\star) = \star, \quad \widetilde{\Phi}^1_{\star,\star}(F) = \Phi \circ F \circ \Phi^{-1},$$

$$\widetilde{\Phi}^1_{\star,\star}\Big(\begin{tikzcd} \star \arrow[r, bend left=50, "F_2"] \arrow[r, bend right=50, "F_1"'] \arrow[r, phantom, "\varphi\Uparrow"] & \star \end{tikzcd}\Big) = \begin{tikzcd} \star \arrow[r, bend left=50, "\Phi^{-1}"] \arrow[r, bend right=50, "\Phi^{-1}"'] \arrow[r, phantom, "Id\Uparrow"] & \star \arrow[r, bend left=50, "F_2"] \arrow[r, bend right=50, "F_1"'] \arrow[r, phantom, "\varphi\Uparrow"] & \star \arrow[r, bend left=50, "\Phi"] \arrow[r, bend right=50, "\Phi"'] \arrow[r, phantom, "Id\Uparrow"] & \star \end{tikzcd} .$$

The 2-isomorphism $\widetilde{\Phi}^2_\star$ and the compatibilities $\widetilde{\Phi}^2_{\star,\star,\star}$ utilise the natural isomorphism between $\Phi^{-1} \circ \Phi$ (or $\Phi \circ \Phi^{-1}$) and the corresponding identity functors. Thus, $\widetilde{\Phi}$ is neither strict, nor unital, in general. We say that two $\mathcal{K}$-algebras $A$ and $B$ are *$\mathcal{K}$-Morita equivalent* if there exist quasiinverse Morita equivalences $\Phi$ and $\Phi^{-1}$ with some choice of a strong module structure such that the homomorphisms of 2-groups $\widetilde{\Phi} \circ \Theta_A, \Theta_B : \widetilde{\mathcal{K}} \to 2\text{-}\mathcal{A}ut(B\text{-Mod})$ are equivalent.

Recall that a Morita context $(A, B, {}_AM_B, {}_BN_A, \alpha, \beta)$ is *nondegenerate* if $\alpha$ and $\beta$ are isomorphisms. We say the Morita context is *$\mathcal{K}$-equivariant* if $A$ and $B$ are $\mathcal{K}$-algebras and

(1) both $M$ and $N$ are $G$-modules,

(2) the bimodule actions $A \otimes_{\mathbb{K}} M \otimes_{\mathbb{K}} B \to M$ and $B \otimes_{\mathbb{K}} N \otimes_{\mathbb{K}} A \to N$ are homomorphisms of $G$-modules,

(3) the bimodule maps $\alpha : M \otimes_B N \to A$ and $\beta : N \otimes_A M \to B$ are homomorphisms of $G$-modules,

(4) ${}^{\partial(h)}m = \omega_A(h) m \omega_B(h)^{-1}$ and ${}^{\partial(h)}n = \omega_B(h) n \omega_A(h)^{-1}$ for all $h \in H$, $n \in N$, $m \in M$.

Observe that axiom (2) manifests in the identities ${}^{\mathbf{g}}(amb) = {}^{\mathbf{g}}a\,{}^{\mathbf{g}}m\,{}^{\mathbf{g}}b$. The following theorem characterises $\mathcal{K}$-Morita equivalences within the context of Morita theory.

**Theorem 2.6.** *The following statements about associative $\mathcal{K}$-algebras $A$ and $B$ are equivalent:*

*(1) The 2-modules $\Theta_A$ and $\Theta_B$ for $\mathcal{K}$ are equivalent.*

*(2) $A$ and $B$ are $\mathcal{K}$-Morita equivalent.*

*(3) There exists a nondegenerate $\mathcal{K}$-equivariant Morita context $(A, B, {}_AM_B, {}_BN_A, \alpha, \beta)$.*

*Proof. (1)⇒(2):* Let $\psi : \Theta_A \Rightarrow \Theta_B$ be a natural 2-isomorphism, $\phi : \Theta_B \Rightarrow \Theta_A$ its quasiinverse. Then $\psi^1_\star : A\text{-Mod} \to B\text{-Mod}$ and $\phi^1_\star : B\text{-Mod} \to A\text{-Mod}$ are quasiinverse category equivalences. Let $\Phi := \psi_\star$,

$\Phi^{-1} := \phi_\star$. We have three 2-functors and two 2-isomorphisms between them:

$$\widetilde{\Phi} \circ \Theta_A \cong \Theta_A \cong \Theta_B \ : \ \widetilde{\mathcal{K}} \to \mathit{2}\text{-}\mathcal{M}od^{\mathbb{K}}.$$

Hence, $\widetilde{\Phi} \circ \Theta_A$ and $\Theta_B$ are equivalent as 2-functors $\widetilde{\mathcal{K}} \to \mathit{2}\text{-}\mathcal{M}od^{\mathbb{K}}$. When we consider 2-$\mathcal{A}ut(B\text{-Mod})$ as a 2-subcategory of $\mathit{2}\text{-}\mathcal{M}od^{\mathbb{K}}$, it contains all invertible 1-objects and 2-objects, related to the 0-object $B$-Mod. Consequently, $\widetilde{\Phi} \circ \Theta_A$ and $\Theta_B$ are equivalent as 2-functors $\widetilde{\mathcal{K}} \to$ 2-$\mathcal{A}ut(B\text{-Mod})$, that is, $A$ and $B$ are $\mathcal{K}$-Morita equivalent.

*(3)⇒(1):* A nondegenerate $\mathcal{K}$-equivariant context $(A, B, {}_AM_B, {}_BN_A, \alpha, \beta)$ gives quasiinverse 2-morphisms $\phi : \Theta_A \to \Theta_B$ and $\psi = \phi^{-1}$ of 2-modules. We define

$$\phi^1_\star : A\text{-Mod} \to B\text{-Mod},\ \phi^1_\star(P) = N \otimes_A P, \ \ \psi^1_\star : B\text{-Mod} \to A\text{-Mod},\ \psi^1_\star(T) = M \otimes_B T.$$

Let us now discuss in detail the natural transformation

$$\phi^2_{\star,\star} : \Theta^1_{A\ \star,\star} \diamond \phi^1_\star \Rightarrow \phi^1_\star \diamond \Theta^1_{B\ \star,\star} \ \text{ where } \ \Theta^1_{A\ \star,\star} \diamond \phi^1_\star,\ \phi^1_\star \diamond \Theta^1_{B\ \star,\star} : \widetilde{\mathcal{K}}_1(\star,\star) \to \mathit{2}\text{-}\mathcal{M}od^{\mathbb{K}}_1(A\text{-Mod}, B\text{-Mod})$$

and $\psi^2_{\star,\star}$ is defined similarly. The objects of the category $\widetilde{\mathcal{K}}_1(\star,\star)$ is the group $G$. Thus, the natural transformation $\phi^2_{\star,\star}$ is defined for all $A$-modules $P$ and $\mathbf{g} \in G$:

$$\phi^2_{\star,\star}(\mathbf{g})_P : N \otimes_A (P^{[\mathbf{g}]}) \to (N \otimes_A P)^{[\mathbf{g}]}, \quad n \otimes p \mapsto {}^{\mathbf{g}}n \otimes p.$$

Let us verify that it is a homomorphism of $B$-modules:

$$\phi^2_{\star,\star}(\mathbf{g})_P(bn \otimes p) = {}^{\mathbf{g}}(bn) \otimes p = {}^{\mathbf{g}}b\ {}^{\mathbf{g}}n \otimes p = b \cdot^{[\mathbf{g}]} ({}^{\mathbf{g}}n \otimes p) = b \cdot^{[\mathbf{g}]} \phi^2_{\star,\star}(\mathbf{g})_P(n \otimes p)$$

for all $b \in B$, $n \in N$, $p \in P$. The fact that $\phi^2_{\star,\star}$ is a natural transformation is equivalent to commutativity of the following diagrams

$$\begin{tikzcd} N \otimes_A P^{[\mathbf{f}]} \arrow[r, "\phi^2_{\star,\star}(\mathbf{f})_P"] & (N \otimes_A P)^{[\mathbf{f}]} \\ N \otimes_A P^{[\mathbf{g}]} \arrow[r, "\phi^2_{\star,\star}(\mathbf{g})_P"] \arrow[u, "\mathrm{Id}_N \otimes h"] & (N \otimes_A P)^{[\mathbf{g}]} \arrow[u, "h"'] \end{tikzcd} \quad \text{for each 2-morphism} \quad \begin{tikzcd} \star \arrow[r, bend left=60, "\mathbf{f}"] \arrow[r, bend right=60, "\mathbf{g}"'] \arrow[r, phantom, "\Uparrow h"] & \star \end{tikzcd} .$$

An element $n \otimes p \in N \otimes_A P^{[\mathbf{g}]}$ is mapped via the bottom-right corner to

$$n \otimes p \mapsto {}^{\mathbf{g}}n \otimes p \mapsto \omega_B(h)\, {}^{\mathbf{g}}n \otimes p$$

and via the top-left corner to

$$n \otimes p \mapsto n \otimes \omega_A(h) \cdot p \mapsto {}^{\mathbf{f}}n \otimes \omega_A(h) \cdot p = {}^{\partial(h)}({}^{\mathbf{g}}n)\omega_A(h) \otimes p$$

that are equal by axiom (4) of an $\mathcal{K}$-equivariant Morita context. There are still two compatibility conditions to verify. The second compatibility condition follows easily from the fact that $\phi^2_{\star,\star}(1_G)_P$ is the identity operator. The first compatibility condition is checked for $\mathbf{a},\mathbf{b}\in G$:

$$\phi_{\mathbf{a},P} : (N\otimes_A P^{[\mathbf{a}]})^{[\mathbf{b}]} \to (N\otimes_A P)^{[\mathbf{a}][\mathbf{b}]},\ \ n\otimes p \mapsto {}^{\mathbf{a}}n\otimes p\ ,$$

$$\phi_{\mathbf{b},P} : N\otimes_A P^{[\mathbf{a}][\mathbf{b}]} \to (N\otimes_A P^{[\mathbf{a}]})^{[\mathbf{b}]},\ \ n\otimes p \mapsto {}^{\mathbf{b}}n\otimes p\ ,$$

$$\phi_{\mathbf{a},\mathbf{b},P} : N\otimes_A P^{[\mathbf{a}][\mathbf{b}]} \to (N\otimes_A P)^{[\mathbf{a}][\mathbf{b}]},\ \ n\otimes p \mapsto {}^{(\mathbf{ab})}n\otimes p.$$

Now the first compatibility boils down to the left action axiom ${}^{(\mathbf{ab})}n = {}^{\mathbf{a}}({}^{\mathbf{b}}n)$.

We can finish the proof at this point because $\phi$ and $\psi$ are quasiinverse 2-morphisms.

*(2)⇒(3):* A $\mathcal{K}$-Morita equivalence is a $G$-Morita equivalence. In particular, it is given (up to natural transformations) by a $G$-equivariant Morita context $(A, B, {}_AM_B, {}_BN_A, \alpha, \beta)$ [9, Theorem 2.3], i.e., a Morita context satisfying axioms (1)–(3) of the definition of a $\mathcal{K}$-equivariant Morita context.

It remains to establish axiom (4). An analogous property ${}^{\partial(h)}a = \omega_A(h)a\omega_A(h)^{-1}$ holds for $a\in A$, $h\in H$. Let us explain briefly how this property gives axiom (4) for $N=\Phi(A)$, leaving technical verifications to an inquisitive reader. The $\mathcal{K}$-Morita equivalence $\Phi$ gives an equivalence of 2-modules $\Theta_A$ and $\Theta_B$. Let us now express the property as an identity on some structure morphisms $A\to A^{[\partial(h)]}$:

$$\partial(h)_A \ = \ \Theta^1_{A\,\star,\star}(1_G \stackrel{h}{\Longrightarrow} \partial(h))_A \circ R(\omega_A(h^{-1}))$$

where $\partial(h)_A : A\to A^{[\partial(h)]}$ is the group action and $R(\omega_A(h^{-1})) : A\to A$ is the $A$-module endomorphism given by the right multiplication. The $\mathcal{K}$-Morita equivalence $\Phi$ preserves this identity: the equality

$$\partial(h)_N \ = \ \Theta^1_{N\,\star,\star}(1_G \stackrel{h}{\Longrightarrow} \partial(h))_N \circ R(\omega_A(h^{-1})) \in \hom(N, N^{[\partial(h)]})$$

holds and it is exactly axiom (4). □

## 3. Structure of 2-representations

There is a critical difference between a 2-module and a 2-representation of $\widetilde{\mathcal{K}}$. An action of $G$ on $A$-Mod for an arbitrary associative algebra $A$ could be hard to pinpoint: thanks to Theorem 2.6 we need to compute the group of invertible $A$-$A$-bimodules, so called the non-commutative Picard group of $A$. However, if $A$ is semisimple, these bimodules could be tracked modulo *realizability* of cocycles. Let $Z=\mathbb{K}^n$ with some action of $G$. We call a cocycle $\mu\in Z^2(G,Z^\times)$ *realizable* if there exists a projective $Z$-semilinear representation of $G$ on a finite dimensional faithful $Z$-module $M$ with cocycle $\mu$:

$$\rho : G\to \mathrm{GL}({}_{\mathbb{K}}M),\ \ \rho(\mathbf{g})(zm) = (\mathbf{g}\cdot z)\rho(\mathbf{g})(m),\ \ \rho(\mathbf{fg}) = \mu(\mathbf{f},\mathbf{g})\rho(\mathbf{f})\rho(\mathbf{g})\ \ \text{for all}\ \ \mathbf{f},\mathbf{g}\in G, z\in Z, m\in M.$$

If $G$ is finite, all cocycles are realizable because projective representations are representations of a finite dimensional twisted skew group algebra $Z_\mu G$ (where $z\mathbf{g} \cdot z'\mathbf{h} = z\,{}^{\mathbf{g}}z' \mu^{-1}(\mathbf{g},\mathbf{h})\mathbf{gh}$). If $G$ is infinite, it is no longer the case. Suppose $G$ is simple, $\mu$ is non-trivial. If $\mu$ is realizable, then the action of $G$ on $A \coloneqq \mathrm{End}_Z M$ is non-trivial, hence faithful. But $G$ could be of higher cardinality than $\mathbb{K}$ or non-linear. In both case, non-trivial cocycles are not realizable.

Let $e_1, \ldots, e_n$ be all primitive idempotents in $Z$, $G_i \leq G$ – the stabiliser of $e_i$. We decompose the cocycle into components $\mu = (\mu_i) = \sum_i \mu_i e_i$, the components $\mu_i$ are cocycles for $G_i$, not the full $G$. Let us collect basic facts about realizability:

**Lemma 3.1.**
1. *$\mu$ is realizable if and only if all $\mu_i|_{G_i^2}$ are realizable.*
2. *If $\mu \sim \nu$, then $\mu$ is realizable if and only if $\nu$ is realizable.*
3. *Realizable cocycles form a subgroup $Z^2_{\mathrm{re}}(G, Z^\times) \leq Z^2(G, Z^\times)$.*

*Proof.* **1.** The only if statement is obvious.

In the opposite direction, the subgroup $H = \cap_i G_i$ is of finite index in $G$ and all cocycles $\mu_i|_{H^2}$ are realizable via projective representations $V_i$. Then $V \coloneqq \oplus_i V_i$ realizes the cocycle $\mu|_{H^2}$, while $Z_\mu G \otimes_{Z_\mu H} V$ realizes $\mu$ itself.

**2.** This is easy to verify for $Z = \mathbb{K}$. The general statement follows from 1.

**3.** Tensor products and contragradient representations verify this for $Z = \mathbb{K}$. The general statement follows from 1. □

It is known (albeit in a different language) that a 2-representation of a finite group $G$ comes from a split semisimple $G$-algebra $A$ (cf. [17] and [9, Theorem 4.3]). To state a general statement covering infinite groups we need *a semimatrix algebra*, by which we mean a finite direct sum of $\mathrm{End}_{\mathbb{K}} V$ for some, not necessarily finite dimensional vector spaces $V$.

**Lemma 3.2.** *(cf. [7, Th. 5.5]) The following statements hold for a 2-representation $R$ of a group $G$.*

1. *The 2-representation $R$ admits an associated cocycle $\mu \in Z^2(G, Z^\times)$ where $Z = \mathbb{K}^n$, $n = R^0(\star)$.*
2. *The cohomology class $[\mu]$ is canonically associated to $R$.*
3. *If $\mu$ is realizable, then there exists a split semisimple $G$-algebra $A$ such that $R$ and $\Theta_A$ are equivalent.*
4. *In general, there exists a semimatrix $G$-algebra $A$ such that $R$ is equivalent to a sub-2-representation of the 2-module $\Theta_A$.*

*Proof.* Using Theorem 1.2, interpret $R$ as a 2-functor $G \to \mathit{2}\text{-}\mathcal{V}ect^{\mathbb{K}}$. Let $L_1, L_2 \ldots L_n$ be all simple non-isomorphic objects of the category $\mathfrak{C} = R^0(\star)$. Without loss of generality, $\mathfrak{C} = Z$-Mod while $R$ is given by the algebraic datum as in Theorem 1.2 (we use $\otimes \coloneqq \otimes_{\mathbb{K}}$ as well as $\otimes_Z$):

$$MR^1_{\star,\star}(\mathbf{g}) = \bigoplus_{i,j} L_j \otimes \hom_Z(L_i, M) \otimes V_{i,j}(\mathbf{g}), \qquad R^2_\star = (\gamma_i \delta_{i,j}), \quad \gamma_i : \mathbb{K} \xrightarrow{\cong} V_{i,i}(1).$$

An element $\mathbf{g} \in G$ yields a permutation $\sigma = \sigma(\mathbf{g}) \in S_n$ so that $L_i R^1(\mathbf{g}) \cong L_{\sigma(i)}$.

**1.** Let us choose bases in all one-dimensional spaces constituting $R^1(\mathbf{g})$. The choice for $R^1(1)$ is canonical, while the rest are not:

$$b_i = b_i(\mathbf{g}) \in V_{i,\sigma(i)}(\mathbf{g}), \quad b_i(1) = \gamma_i(1) \in V_{i,i}(1).$$

The collection of 2-isomorphisms $R^2(\mathbf{f},\mathbf{g})$ yields a 2-cocycle $\mu \in Z^2(G, Z^\times)$ whose components $\mu_i(\mathbf{g},\mathbf{h})$ (where $\mathbf{g},\mathbf{h} \in G$, $i = \sigma(\mathbf{gh})(j) = \sigma(\mathbf{h})(k)$) are defined as a composition (notice $\mathrm{End}_Z L_i = \mathbb{K}$):

$$L_i \xrightarrow{v\mapsto v\otimes b_j(\mathbf{g})\otimes b_k(\mathbf{h})} (L_j R^1(\mathbf{g}))R^1(\mathbf{h}) = L_i \otimes V_{j,k}(\mathbf{g}) \otimes V_{k,i}(\mathbf{h}) \xrightarrow{R^2(\mathbf{g},\mathbf{h})} L_j R^1(\mathbf{gh}) = L_i \otimes V_{j,i}(\mathbf{gh}) \xrightarrow{v\otimes b_j(\mathbf{gh})\mapsto v} L_i.$$

The cocycle condition follows from the pentagon condition for $R^2_{\star,\star,\star}$.

**2.** A different choice of elements $b_i(\mathbf{g})$ yields a cohomologous cocycle.

**3.** Let us realize the cocycle $\mu^{-1}$ on a finite-dimensional $Z$-module $S$. If $S_i = e_i S$ is a Peirce decomposition of $S$, then we get a split semisimple $G$-algebra $A \coloneqq \mathrm{End}_Z S = \oplus_i \mathrm{End}_{\mathbb{K}} S_i$. Let $L \coloneqq \oplus_i L_i$. Let us define an equivalence of 2-representations $\psi : R \to \Theta_A$:

$$\psi^1_\star(M) \coloneqq \hom_Z(L, M) \otimes_Z S = \bigoplus_i \hom_Z(L_i, M) \otimes S_i,$$

$$\psi^2_{\star,\star\ M,\mathbf{g}} : \bigoplus_{i,j} S_j \otimes \hom_Z(L_i, M) \otimes V_{i,j}(\mathbf{g}) \to \bigoplus_i S_i^{[\mathbf{g}]} \otimes \hom_Z(L_i, M),\ s \otimes \phi \otimes b_i(\mathbf{g}) \mapsto {}^{\mathbf{g}}s \otimes \phi.$$

While the quasi-invertibility of $\psi$ is apparent, we need to check the two coherence conditions from Section 1. The components of the first coherence condition are computed for any pair $\mathbf{f},\mathbf{g} \in G$. It is convenient to think of $V(\mathbf{g}) = \oplus_{i,j} V_{i,j}(\mathbf{g})$ as a $Z-Z-$bimodule with a generator $b(\mathbf{g}) = \sum_i b_i(\mathbf{g})$ for this computation:

$$\psi_{\mathbf{f},\mathbf{g}} : (\hom_Z(L, M) \otimes_Z S) \otimes_Z V(\mathbf{f}) \otimes_Z V(\mathbf{g}) \to \hom_Z(L, M) \otimes_Z S^{[\mathbf{fg}]},\ \ \phi \otimes s \otimes b(\mathbf{f}) \otimes b(\mathbf{g}) \mapsto \mu(\mathbf{f},\mathbf{g})^{-1} \phi \otimes {}^{\mathbf{fg}}s,$$

$$\psi_{\mathbf{g}} : (\hom_Z(L, M) \otimes_Z S) \otimes_Z V(\mathbf{f}) \otimes_Z V(\mathbf{g}) \to (\hom_Z(L, M) \otimes_Z S^{[\mathbf{g}]}) \otimes_Z V(\mathbf{f}),\ \ \phi \otimes s \otimes b(\mathbf{f}) \otimes b(\mathbf{g}) \mapsto \phi \otimes b(\mathbf{f}) \otimes {}^{\mathbf{g}}s,$$

$$\psi_{\mathbf{f}} : (\hom_Z(L, M) \otimes_Z S^{[\mathbf{g}]}) \otimes_Z V(\mathbf{f}) \to \hom_Z(L, M) \otimes_Z S^{[\mathbf{fg}]},\ \ \phi \otimes s \otimes b(\mathbf{f}) \mapsto \phi \otimes {}^{\mathbf{f}}s.$$

The equality $\psi_{\mathbf{f},\mathbf{g}} = \psi_{\mathbf{f}} \circ \psi_{\mathbf{g}}$ is precisely the cocycle condition for the projective $G$-module ${}_Z S$. The second coherence condition degenerates since $\mathbf{i}_{\mathcal{C}}$ and $\Theta^2_{A\,\star}$ are identities. The coherence condition becomes

$$\psi^1_\star(M) \xrightarrow{R^{1\ -1}_\star} \psi^1_\star(R^1(1) \diamond M) \xrightarrow{\psi^2_{\star,\star}(1)} \psi^1_\star(M) \diamond \Theta^1_{A\ \star,\star}(1) \xrightarrow{=} \psi^1_\star(M),$$

once applied to an object $M \in \mathcal{C}$. This is equal to the identity:

$$\phi \otimes s \xrightarrow{R^{1\ -1}_\star} \phi \otimes s \otimes b(1) \xrightarrow{\psi^2_{\star,\star}(1)} {}^1 s \otimes \phi \xrightarrow{=} \phi \otimes s.$$

**4.** If $\mu$ is not necessarily realizable, the argument in Part 3 still works. The only difference is that $S$ is infinite dimensional. This means that $S_i$ is infinite dimensional for some $i$ and the corresponding direct summand $\mathrm{End}_Z S_i$ is an infinite full matrix algebra. Consequently, the image of $\psi$ in $\Theta_A$ consists of those semisimple $A$-modules that are direct sums of standard modules. (Notice that the infinite full matrix algebra has also non-standard simple modules.) $\square$

This result extends to 2-groups using Extension Lemma 2.4.

**Corollary 3.3.** *Consider a 2-representation $R$ of $\widetilde{\mathcal{K}}$ such that its restriction to $G$ has a realizable cocycle $\mu$. Then there exists a split semisimple $\mathcal{K}$-algebra $A$ such that $R$ and $\Theta_A$ are equivalent. Furthermore, 2-representations $\Theta_A$ and $\Theta_B$ are equivalent if and only if $A$ and $B$ are $\mathcal{K}$-Morita equivalent.*

*Proof.* Lemma 3.2 gives a split semisimple $G$-algebra $A$. Lemma 2.5 makes the 2-representation unital. Lemma 2.4 extends the $G$-algebra structure on $A$ to a $\mathcal{K}$-algebra structure. Theorem 2.6 establishes the second statement. $\square$

Let us call (weak) 2-representation of $\widetilde{\mathcal{K}}$, described in Corollary 3.3, *realizable*. To deal with not necessarily realizable weak 2-representations, we need to contemplate a semimatrix algebra $A = \oplus_i \mathrm{End}_{\mathbb{K}} S_i$ and its full subcategory $A$-Mod$^\circ$ of $A$-Mod consisting of semisimple representations whose direct summands are isomorphic to $S_i^{[\mathbf{g}]}$ for some automorphisms $\mathbf{g}$.

If $A$ is a semimatrix $\mathcal{K}$-algebra, we can describe the structure of the 2-representation $R = \Theta_A^\circ$ of $\widetilde{\mathcal{K}}$ afforded by $A$ in the same way as in the proof of Theorem 1.2. Notice that $\Theta_A^\circ = \Theta_A$ for a split semisimple algebra $A$. Let us depict it using the bicategory 2-Vect$^{\mathbb{K}}$ for the benefit of the reader:

$$R^0(\star) = n, \quad R^1(\mathbf{g})_{i,j} = \hom(S_i, S_j^{[\mathbf{g}]}), \quad R^2_{\star\ i,i} : \hom(S_i, S_i) \xrightarrow{\cong} \mathbb{K},$$

$$R^2(\mathbf{f}, \mathbf{g}) : \bigoplus_j R^1(\mathbf{g})_{i,j} \otimes R^1(\mathbf{f})_{j,k} \to R^1(\mathbf{gf})_{i,k}, \quad \sum_j \varphi_{i,j} \otimes \psi_{j,k} \mapsto \sum_j \psi_{j,k} \circ \varphi_{i,j},$$

$$R^1(\mathbf{g}, h) = R^1(\mathbf{g} \overset{h}{\Longrightarrow} \partial h\, \mathbf{g}) : \hom(S_i, S_j^{[\mathbf{g}]}) \to \hom(S_i, S_j^{[\partial(h)\mathbf{g}]}), \quad : \varphi \mapsto \omega_A(h) \circ \varphi.$$

It is useful to get a version of Corollary 3.3 for semimatrix algebras. We need a version of Lemma 2.1:

**Lemma 3.4.** *Suppose* $\mathbf{f}$ *and* $\mathbf{g}$ *are automorphisms of a semimatrix algebra $A$,* $[\mathbf{f}]$, $[\mathbf{g}]$ *are twists of the category on* $A$-Mod$^\circ$. *Then the map (cf. Lemma 2.1)*

$$\Upsilon : \{x \in A \mid \forall a \in A \ \ xa = \mathbf{g}(\mathbf{f}^{-1}(a))x\} \to \mathrm{Nat.Trans}([\mathbf{f}], [\mathbf{g}]), \quad \Upsilon(x)_M : m \mapsto x \cdot m$$

*is a bijection.*

*Proof.* Thanks to Lemma 2.1, we just need to construct an inverse map $\Xi^\circ$. Given a natural transformation $\varphi$, its value $\varphi_i \coloneqq \varphi_{S_i}$ is a linear map $S_i \to S_i$. Hence, $\varphi_i \in \mathrm{End}_{\mathbb{K}} S_i \subseteq A$ and the inverse map is

$$\Xi^\circ : \mathrm{Nat.Trans}([\mathbf{f}], [\mathbf{g}]) \to \{x \in A \mid \forall a \in A \ \ xa = \mathbf{g}(\mathbf{f}^{-1}(a))x\}, \quad \Xi^\circ(\varphi) = \sum_i \varphi_i.$$

$\square$

Now we can prove a version of Corollary 3.3 for non-realizable cocycles. It lacks uniqueness statement.

**Corollary 3.5.** *For a 2-representation $R$ of $\widetilde{\mathcal{K}}$ there exists a semimatrix $\mathcal{K}$-algebra $A$ such that $R$ and $\Theta^\circ_A$ are equivalent. In particular, any 2-representation is equivalent to a strict 2-representation.*

*Proof.* Lemma 3.2 yields a semimatrix $G$-algebra $A$ so that $R \cong \Theta^\circ_A$ as 2-representations of $G$. Lemma 2.5 makes the 2-representation unital. Transformations $R^1(\mathbf{g}, x)$ and $R^2(\mathbf{f}, \mathbf{g})$ become natural transformations between twists of $A$-Mod$^\circ$, hence, Lemma 3.4 associates elements of $A$ to all $R^1(\mathbf{g}, x)$ and $R^2(\mathbf{f}, \mathbf{g})$. All the proofs in Lemma 2.4 work for $A$-Mod$^\circ$ for a semimatrix algebra $A$, yielding the required $\mathcal{K}$-algebra structure on $A$. $\square$

Using Corollary 3.3 and Corollary 3.5, we can describe various constructions on realizable 2-representations of $\widetilde{\mathcal{K}}$. It is instructive for the reader to verify that each of the constructions depend only on the 2-representation $R$ but not on its particular realization $R = \Theta_A$ or $R = \Theta^\circ_A$. Tensor products and directs sums of arbitrary 2-representations are defined by Barrett and Mackaay [3]. Our definitions agree with theirs and we follow their notation.

*A direct sum* of $\Theta^\circ_A$ and $\Theta^\circ_B$ comes from the direct sum of algebras:

$$\Theta^\circ_A \boxplus \Theta^\circ_B \coloneqq \Theta^\circ_{A\oplus B}.$$

The $\mathcal{K}$-structure $\omega_{A\oplus B}$ on the direct sum is the obvious one:

$$\omega_{A\oplus B}(\mathbf{g}) = (\omega_A(\mathbf{g}), \omega_B(\mathbf{g})), \quad \omega_{A\oplus B}(h) = \omega_A(h) \oplus \omega_B(h).$$

The degree of $\Theta^\circ_A \boxplus \Theta^\circ_B$ is the sum of the degrees. A similar description for *the tensor product*:

$$\Theta_A \boxtimes \Theta_B \coloneqq \Theta_{A\otimes B}$$

works only for split semisimple algebras. A slight modification is required in general

$$\Theta^\circ_A \boxtimes \Theta^\circ_B \coloneqq \Theta^\circ_{A\widehat{\otimes}B}, \;\; A\widehat{\otimes}B \coloneqq \bigoplus_{i,j} \mathrm{End}_{\mathbb{K}}(S_i \otimes T_j) \;\; \text{if} \;\; A = \bigoplus_i \mathrm{End}_{\mathbb{K}} S_i, \;\; B = \bigoplus_j \mathrm{End}_{\mathbb{K}} T_j \; .$$

Since $A \otimes B$ is a subalgebra of $A\widehat{\otimes}B$, the $\mathcal{K}$-structures $\omega_{A\otimes B}$ and $\omega_{A\widehat{\otimes}B}$ are given by the same obvious formula:

$$\omega_{A\widehat{\otimes}B}(\mathbf{g}) : a \otimes b \mapsto \omega_A(\mathbf{g})(a) \otimes \omega_B(\mathbf{g})(b), \quad \omega_{A\widehat{\otimes}B}(h) = \omega_A(h) \otimes \omega_B(h).$$

The degree of $\Theta^\circ_A \boxtimes \Theta^\circ_B$ is the product of the degrees. *The contragradient 2-representation* of $\Theta_A$ can be defined using the opposite multiplication algebra $A^{op}$:

$$\Theta^*_A \coloneqq \Theta_{A^{op}}, \quad \omega_{A^{op}}(\mathbf{g}) = \omega_A(\mathbf{g}), \quad \omega_{A^{op}}(h) = \omega_A(h)^{-1}.$$

It does not work for semimatrix algebra because $A$ has finite columns but $A^{op}$ has finite rows. The contragradient 2-representation can be defined using the dual vector spaces:

$$\Theta^{\circ}_A{}^* \coloneqq \Theta_{A^\sharp}, \;\; A^\sharp \coloneqq \bigoplus_i \mathrm{End}_{\mathbb{K}} S_i^* \;\; \text{if} \;\; A = \bigoplus_i \mathrm{End}_{\mathbb{K}} S_i, \; \omega_{A^\sharp}(\mathbf{g}) = \omega_A(\mathbf{g})^*, \quad \omega_{A^\sharp}(h) = \omega_A(h)^* \; .$$

The following immediate fact will be useful.

**Lemma 3.6.** *The set of equivalence classes of degree one 2-representations* $2\text{-Rep}_1(\mathcal{K})$ *is an abelian group under the tensor product* $\boxtimes$*. The inverse is given by the contragradient representation. Realizable degree one 2-representations form a subgroup.*

*A sub-2-representation* of $\Theta^\circ_A$ is $\Theta^\circ_B$ where $A$ is split into a direct sum of ideals $A = B \oplus C$ such that $B$ is $G$-stable (observe that $C$ is $G$-stable automatically). The group $G$ acts on $B$ and the group $H$ is mapped to $B$ by the projection to the first factor: $\omega_B : H \xrightarrow{\omega_A} A \xrightarrow{\pi_1} B$. A 2-representation $\Theta^\circ_A$ is *irreducible* if no such non-trivial splitting exists.

Let $\mathcal{K}' = (H' \to G')$ be a crossed submodule of $\mathcal{K} = (H \to G)$, i.e. $G'$ is a subgroup of $G$ and $H'$ is a $G'$-stable subgroup of $H$ such that $\partial(H') \subseteq G'$. Then *the restriction* of $\Theta^\circ_A$ is just $\Theta^\circ_A$ considered as a 2-representation of $\mathcal{K}'$.

We define the adjoint functor *induction* only under two special assumptions:

$$H' = H \quad \text{and} \quad |G : G'| \text{ is finite.}$$

Let $A$ be a semimatrix $\mathcal{K}'$-algebra. The finite $G$-set $G/G'$ has a sheaf of algebras $G \times_{G'} A$. Its space of global sections

$$\widetilde{A} \coloneqq \Gamma(G/G', G \times_{G'} A) = \oplus_{\mathbf{g}\in G}(\mathbf{g}, A)/\sim$$

is an algebra. Each $(\mathbf{g}, A)$ is just the algebra $A$. The direct sum $\oplus_{\mathbf{g}\in G}(\mathbf{g}, A)$ is the usual direct sum of algebras. The congruence $\sim$ is defined by

$$(\mathbf{gh}, a) \sim (\mathbf{g}, {}^{\mathbf{h}}a) \text{ for all } \mathbf{g} \in G, \mathbf{h} \in G', a \in A.$$

Hence, a left transversal $T$ to $G'$ in $G$ gives a presentation

$$\widetilde{A} = \oplus_{\mathbf{t}\in T}(\mathbf{t}, A)$$

as a direct sum of ideals, i.e., $\widetilde{A}$ is a semimatrix algebra as each $(\mathbf{t}, A)$ is just $A$. We define the structure of a $\mathcal{K}$-algebra on $\widetilde{A}$:

$${}^{\mathbf{g}}(\mathbf{t}, a) \coloneqq (\mathbf{gt}, a) = (\mathbf{t}', {}^{\mathbf{g}'}a), \quad \omega_{\widetilde{A}}(h) = \sum_{\mathbf{t}\in T}(\mathbf{t}, \omega_A({}^{\mathbf{t}^{-1}}h))$$

where $h \in H$, $\mathbf{g} \in G$ and $\mathbf{gt} = \mathbf{t}'\mathbf{g}'$ for unique $\mathbf{t}' \in T$. Let us verify the axiom:

$${}^{\partial(h)}(\mathbf{t}, a) = (\mathbf{t}(\mathbf{t}^{-1}\partial(h)\mathbf{t}), a) = (\mathbf{t}\partial({}^{\mathbf{t}^{-1}}h), a) = (\mathbf{t}, {}^{\partial({}^{\mathbf{t}^{-1}}h)}a) = (\mathbf{t}, \omega_A({}^{\mathbf{t}^{-1}}h)\, a\, \omega_A(({}^{\mathbf{t}^{-1}}h)^{-1})) = \omega_{\widetilde{A}}(h)(\mathbf{t}, a)\omega_{\widetilde{A}}(h^{-1}).$$

Finally, we define $\Theta^\circ_A \uparrow^{\widetilde{\mathcal{K}}}_{\widetilde{\mathcal{K}}'} \coloneqq \Theta^\circ_{\widetilde{A}}$. We finish this section with classification of 2-representations.

**Theorem 3.7.** *Let* $\mathcal{K} = (H \to G)$ *be a crossed module.*

1. *If $\Theta$ is a 2-representation of $\widetilde{\mathcal{K}}$, then there exist irreducible 2-representations $\Theta_1, \dots \Theta_n$ such that $\Theta \cong \Theta_1 \boxplus \Theta_2 \boxplus \dots \Theta_n$.*
2. *The 2-representations $\Theta_1, \dots \Theta_n$ in 1. are unique up to a permutation and equivalence.*
3. *If $\Theta$ is an irreducible 2-representation of $\widetilde{\mathcal{K}}$, then there exists a subgroup of finite index $G'$ such that $\partial(H) \subseteq G' \subseteq G$ and a degree one 2-representation $\Psi$ of $\widetilde{\mathcal{K}'}$ where $\mathcal{K}' = (H \to G')$ is a crossed submodule such that $\Theta \cong \Psi \uparrow_{\widetilde{\mathcal{K}'}}^{\widetilde{\mathcal{K}}}$*
4. *The pair $(G', \Psi)$ in 3. is unique up to conjugation by an element of $G$.*

*Proof.* **1.** Without loss of generality, $\Theta = \Theta_A^\circ$ where $A = \oplus_{i=1}^m A_i$ where $A_i$ are full matrix algebras. Let $X_1, \dots X_n$ be the $G$-orbits on the set $\{1, \dots m\}$, $B_j := \oplus_{i \in X_j} A_i$. Then $\Theta_j := \Theta_{B_j}^\circ$ is an irreducible 2-representation and $\Theta \cong \Theta_1 \boxplus \Theta_2 \boxplus \dots \Theta_n$.

**2.** Suppose $\Theta \cong \Theta_1 \boxplus \Theta_2 \boxplus \dots \Theta_n \xrightarrow{\psi \ \cong} \Theta_1' \boxplus \Theta_2' \boxplus \dots \Theta_{n'}'$. Then $\psi_\star^1 : \{1, \dots m\} \to \{1, \dots m'\}$ is an isomorphism of $G$-sets. $G$-orbits on $\{1, \dots m\}$ come from the direct summands $\Theta_1, \Theta_2, \dots \Theta_n$, while $G$-orbits on $\{1, \dots m'\}$ come from the direct summands $\Theta_1', \Theta_2', \dots \Theta_{n'}'$. Whenever $\psi_\star^1$ moves the $G$-orbit corresponding to $\Theta_i$ to the $G$-orbit corresponding to $\Theta_j'$, the equivalence $\psi$ restricts to an isomorphism $\Theta_i \cong \Theta_j'$.

**3.** In this case the set $\{1, \dots m\}$ consists of a single $G$-orbit. Let $G'$ be the stabiliser of 1 in this set. Then $\Psi := \Theta_{A_1}^\circ$ is a degree one 2-representation of $\widetilde{\mathcal{K}'}$ and $\Theta \cong \Psi \uparrow_{\widetilde{\mathcal{K}'}}^{\widetilde{\mathcal{K}}}$

**4.** In another pair $(\widetilde{G}, \widetilde{\Psi})$ the group $\widetilde{G}$ is a stabiliser of some $k \in \{1, \dots m\}$. Let $\mathbf{g} \in G$ be such that ${}^{\mathbf{g}}1 = k$. Then $\widetilde{G} = \mathbf{g} G' \mathbf{g}^{-1}$ and the 2-representations $\Psi$ and $\mathbf{g}^{-1} \Psi \mathbf{g}$ of $\widetilde{\mathcal{K}'}$ are equivalent. □

## 4. Burnside ring of a 2-group

Let us consider a crossed module $\mathcal{K} = (H \xrightarrow{\partial} G)$ such that $\pi_1(\mathcal{K})$ is a finite group. Let $\mathcal{S}(\mathcal{K})$ be the category of subgroups of $\pi_1(\mathcal{K})$ (cf. [9]). Objects of $\mathcal{S}(\mathcal{K})$ are subgroups of $\pi_1(\mathcal{K})$. The morphisms $\mathcal{S}(P, Q)$ are conjugations $\gamma_{\mathbf{x}} : P \to Q$, $\gamma_{\mathbf{x}}(\mathbf{a}) = \mathbf{x}\mathbf{a}\mathbf{x}^{-1}$, $\mathbf{x} \in \pi_1(\mathcal{K})$ whenever $\mathbf{x} P \mathbf{x}^{-1} \subseteq Q$, restricted to $P$. If $\mathbf{y}^{-1}\mathbf{x} \neq 1_G$ is in the centraliser of $P$, then $\gamma_{\mathbf{x}}$ and $\gamma_{\mathbf{y}}$ are the same conjugations, but we treat them as different morphisms in $\mathcal{S}(P, Q)$. In this respect we are different from the setup, studied by Gunnells, Rose and Rumynin [9], where these are the same morphism. The setups are not drastically different, so we can still use their results exercising a certain care.

A subgroup $P \leq \pi_1(\mathcal{K})$ has an inverse image $\bar{P} := (G \to \pi_1(\mathcal{K}))^{-1}(P)$. It gives a restricted crossed module $\mathcal{K}_P := (H \xrightarrow{\partial} \bar{P})$ with $\pi_1(\mathcal{K}_P) \cong P$.

Let $\Phi$ be the functor "2-representations of degree one". It is a contravariant functor from $\mathcal{S}(\mathcal{K})$ to the category of abelian groups. On objects, $\Phi(P) := \text{2-Rep}_1(\widetilde{\mathcal{K}_P})$. Let us look at a morphism $\gamma_{\mathbf{x}} : P \to Q$. By picking a lifting $\dot{\mathbf{x}} \in G$, i.e., an element with $\dot{\mathbf{x}}\partial(H) = \mathbf{x}$, we get a conjugation morphism of crossed modules

$$\gamma_{\dot{\mathbf{x}}} : \mathcal{K}_P \to \mathcal{K}_Q, \quad P \ni \mathbf{g} \mapsto \dot{\mathbf{x}}\mathbf{g}\dot{\mathbf{x}}^{-1}, \quad H \ni h \mapsto {}^{\dot{\mathbf{x}}}h$$

and the corresponding homomorphism of 2-groups

$$\gamma_{\dot{\mathbf{x}}} : \widetilde{\mathcal{K}_P} \to \widetilde{\mathcal{K}_Q}, \quad P \ni \mathbf{g} \mapsto \dot{\mathbf{x}}\mathbf{g}\dot{\mathbf{x}}^{-1}, \quad \begin{tikzcd} \star \arrow[r, bend left=50, "\mathbf{g}_2"] \arrow[r, bend right=50, "\mathbf{g}_1"'] \arrow[r, phantom, "h\Uparrow"] & \star \end{tikzcd} \mapsto \begin{tikzcd} \star \arrow[r, bend left=50, "\dot{\mathbf{x}}\mathbf{g}_2\dot{\mathbf{x}}^{-1}"] \arrow[r, bend right=50, "\dot{\mathbf{x}}\mathbf{g}_1\dot{\mathbf{x}}^{-1}"'] \arrow[r, phantom, "{}^{\dot{\mathbf{x}}}h\Uparrow"] & \star \end{tikzcd}.$$

An element $h \in H$ gives another lifting $\partial(h)\dot{\mathbf{x}} \in G$ of $\mathbf{x}$ and a new 2-morphism $\gamma_{\partial(h)\dot{\mathbf{x}}} : \widetilde{\mathcal{K}_P} \to \widetilde{\mathcal{K}_Q}$. In essence, they differ by an inner 2-isomorphism determined by $h$. Let us spell it out.

**Lemma 4.1.** *Let $\Theta$ be a 2-representation of $\widetilde{\mathcal{K}_Q}$. Then the 2-representations $\Theta \circ \gamma_{\dot{\mathbf{x}}}$ and $\Theta \circ \gamma_{\partial(h)\dot{\mathbf{x}}}$ of $\widetilde{\mathcal{K}_P}$ are equivalent.*

*Proof.* By Corollary 3.5 the 2-representation $\Theta$ is equivalent to $\Theta^{\circ}_{A}$. The latter comes from a weak homomorphism of crossed modules $\theta : \mathcal{K}_Q \to 2\text{-Aut}(A)$. Consequently,

$$\theta \circ \gamma_{\partial(h)\dot{\mathbf{x}}}(\mathbf{g}) = \theta(\partial(h)\dot{\mathbf{x}}\mathbf{g}\dot{\mathbf{x}}^{-1}\partial(h)^{-1}) = \theta(\partial(h))[\theta \circ \gamma_{\dot{\mathbf{x}}}(\mathbf{g})]\theta(\partial(h))^{-1} = \theta(h)[\theta \circ \gamma_{\dot{\mathbf{x}}}(\mathbf{g})]\theta(h)^{-1}$$

for $\mathbf{g} \in P$ and

$$\theta \circ \gamma_{\partial(h)\dot{\mathbf{x}}}(g) = \theta({}^{\partial(h)\dot{\mathbf{x}}}g) = \theta(h\,{}^{\dot{\mathbf{x}}}gh^{-1}) = \theta(h)[\theta \circ \gamma_{\dot{\mathbf{x}}}(g)]\theta(h)^{-1}$$

for $g \in H$. $\square$

Lemma 4.1 applies to 2-representations, hence, $\gamma_{\dot{\mathbf{x}}}$ and $\gamma_{\partial(h)\dot{\mathbf{x}}}$ determine the same pull-back homomorphism that allows us to define the functor $\Phi$ on morphisms:

$$\Phi(\gamma_{\mathbf{x}}) \coloneqq [\gamma_{\dot{\mathbf{x}}}] = [\gamma_{\partial(h)\dot{\mathbf{x}}}] : \Phi(Q) = 2\text{-Rep}_1(\mathcal{K}_Q) \to 2\text{-Rep}_1(\mathcal{K}_P) = \Phi(P).$$

In general, if the conjugations $\gamma_{\mathbf{x}}$ and $\gamma_{\mathbf{y}}$ are the same, these pull-backs can be different: $\Theta \circ \gamma_{\dot{\mathbf{x}}}$ and $\Theta \circ \gamma_{\dot{\mathbf{y}}}$ are not necessarily equivalent because the actions of 2-objects of $\widetilde{\mathcal{K}_Q}$ could be different. This necessitates our version of the category $\mathcal{S}(\mathcal{K})$ (cf. [9]).

Despite this slight difference, the functor $\Phi$ still leads to the generalised Burnside ring $\mathbb{B}_{\mathbb{A}}(\mathcal{K}) \coloneqq \mathbb{B}^{\Phi}_{\mathbb{A}}(\pi_1(\mathcal{K}))$ with coefficients in a commutative ring $\mathbb{A}$ [9]. The $\mathbb{A}$-basis of $\mathbb{B}_{\mathbb{A}}(\mathcal{K})$ consists of pairs $\langle \Theta, P \rangle$ where $P$ is a subgroup of $\pi_1(\mathcal{K})$, $\Theta$ is a degree one 2-representation of $\widetilde{\mathcal{K}_P}$. In each $\pi_1(\mathcal{K})$-conjugacy class of such pairs we choose one representative because

$$\langle \Theta, P \rangle = \langle \Phi(\gamma_{\mathbf{x}})(\Theta), \mathbf{x}^{-1}P\mathbf{x} \rangle$$

for all $\mathbf{x} \in \pi_1(\mathcal{K})$. We can also write a pair $\langle \Theta, P \rangle$ with an arbitrary 2-representation $\Theta$ of $\widetilde{\mathcal{K}_P}$ but they can be rewritten as linear combinations of pairs with degree one 2-representations by the formulas

$$\langle \Theta_1 \boxplus \Theta_2, P \rangle = \langle \Theta_1, P \rangle + \langle \Theta_2, P \rangle \text{ and } \langle \Theta \uparrow^{\widetilde{\mathcal{K}_P}}_{\widetilde{\mathcal{K}_Q}}, P \rangle = \langle \Theta, Q \rangle.$$

The multiplication in $\mathbb{B}_{\mathbb{A}}(\mathcal{K})$ is $\mathbb{A}$-bilinear, defined on the basis by the formula

$$\langle \Theta, P\rangle \cdot \langle \Omega, Q\rangle = \sum_{P\mathbf{x}Q\in P\backslash \pi_1(\mathcal{K})/Q} \langle \Phi(\gamma_1 : P\cap \mathbf{x}Q\mathbf{x}^{-1} \to P)(\Theta) \boxtimes \Phi(\gamma_{\mathbf{x}^{-1}} : P\cap \mathbf{x}Q\mathbf{x}^{-1} \to Q)(b), P\cap \mathbf{x}Q\mathbf{x}^{-1}\rangle.$$

Theorem 3.7 together with known arguments [9, Section 1] emanates the following proposition.

**Proposition 4.2.** *Let $K_{\mathbb{A}}(\mathcal{K})$ be the Grothendieck ring of 2-representations of the 2-group $\widetilde{\mathcal{K}}$ with coefficients in a commutative ring $\mathbb{A}$, where the multiplication comes from the tensor product $\boxtimes$. The assignment $[\Theta \uparrow_{\widetilde{\mathcal{K}}_P}^{\widetilde{\mathcal{K}}}] \mapsto \langle \Theta, P\rangle$ extended by $\mathbb{A}$-linearity is an $\mathbb{A}$-algebra isomorphism $K_{\mathbb{A}}(\mathcal{K}) \to \mathbb{B}_{\mathbb{A}}(\mathcal{K})$.*

Let us introduce *a mark homomorphism* [9, Lemma 1.2]. We need to assume that the order $|\pi_1(\mathcal{K})|$ is invertible in $\mathbb{A}$. Let $\alpha : \Phi(P) \to \mathbb{A}^\times$ be a group homomorphism. The corresponding mark is an $\mathbb{A}$-algebra homomorphism $f_P^\alpha : \mathbb{B}_{\mathbb{A}}(\mathcal{K}) \to \mathbb{A}$ given by the formula

$$f_P^\alpha(\langle \Theta, Q\rangle) = \frac{1}{|Q|} \sum_{\mathbf{g}\in X} \alpha(\Phi(\gamma_{\mathbf{g}} : P \to Q)(\Theta))$$

where $X = \{\mathbf{g} \in \pi_1(\mathcal{K}) \mid \mathbf{g}P\mathbf{g}^{-1} \subseteq Q\}$. The marks work magnificently for the ring $\mathbb{B}_{\mathbb{A}}(\mathcal{K})$ if all the groups $\Phi(P)$ are finite [9, Corollary 1.3]:

**Proposition 4.3.** *Suppose all $\Phi(P)$ are finite. Let $N$ be the least common multiple of all the orders of elements in various $\Phi(P)$. If $\mathbb{A}$ is a field, containing a primitive $N$-th root of unity, then the mark homomorphisms define an isomorphism of $\mathbb{A}$-algebras*

$$\mathbb{B}_{\mathbb{A}}(\mathcal{K}) \xrightarrow{\cong} \oplus\mathbb{A} = \mathbb{A}^k$$

*where $k$ is the number of $G$-orbits on the disjoint unions $\dot{\cup}_P \Phi(P)$.*

If $\varphi : \mathcal{K}' \to \mathcal{K}$ is a homomorphism of crossed modules, the pull-back of 2-representations gives a homomorphism of Burnside rings $\varphi^* : \mathbb{B}_{\mathbb{A}}(\mathcal{K}) \to \mathbb{B}_{\mathbb{A}}(\mathcal{K}')$. Consider the quotient homomorphism of crossed modules

$$\varphi : \mathcal{K} \to \bar{\mathcal{K}} := (1 \to \pi_1(\mathcal{K})), \quad G \ni \mathbf{g} \mapsto \mathbf{g}\partial(H), \quad H \ni h \mapsto 1.$$

The Burnside ring $\mathbb{B}_{\mathbb{A}}(\bar{\mathcal{K}})$ is precisely the generalised Burnside ring of $\pi_1(\mathcal{K})$ studied by Gunnells, Rose and Rumynin [9], because there are no non-trivial 2-objects in $\widetilde{\bar{\mathcal{K}}}$. All the groups $\Phi(P) = H^2(P, \mathbb{K}^\times)$ are finite. Proposition 4.3 tells us that if $\mathbb{A}$ is a field, containing a primitive $N$-th root of unity, then the corresponding pull-back algebra homomorphism $\varphi^* : \mathbb{B}_{\mathbb{A}}(\bar{\mathcal{K}}) = \mathbb{A}^k \to \mathbb{B}_{\mathbb{A}}(\mathcal{K})$ is injective. Its image can be thought of as the Grothendieck ring of 2-representations "trivial" on $H$.

## 5. Ganter-Kapranov 2-character

Let us recall the notion of a 2-categorical trace [8]. Let $\mathcal{C}$ be a bicategory, $x \in \mathcal{C}_0$ its 0-object. *The 2-categorical trace* of a 1-morphism $u \in \mathcal{C}_1(x, x)$ is the set $\mathbb{T}\mathrm{r}_x(u) := \mathcal{C}_2(\mathbf{i}_x, u)$. It is instructive to observe that in the bicategory of 2-vector spaces 2-Vect$^{\mathbb{K}}$ a 1-morphism $u = (U_{i,j})$ is an $n \times n$-matrix of vector spaces, while its trace is the vector space

$$\mathbb{T}\mathrm{r}_n(u) = \bigoplus_i \hom_{\mathbb{K}}(\mathbb{K}, U_{i,i}) \oplus \bigoplus_{i \neq j} \hom_{\mathbb{K}}(0, U_{i,j}) \cong \bigoplus_i U_{i,i} \ .$$

Let $\Theta$ be a 2-representation of degree $n$ of the 2-group $\widetilde{\mathcal{K}}$. Pick two elements $\mathbf{a}, \mathbf{b} \in G$ such that their images in the fundamental group commute: $\bar{\mathbf{a}}\bar{\mathbf{b}} = \bar{\mathbf{b}}\bar{\mathbf{a}} \in \pi_1(\mathcal{K})$ and $h \in H$ such that $\partial(h)\mathbf{a}\mathbf{b}\mathbf{a}^{-1} = \mathbf{b}$. This data gives a linear operator $\mathbb{X}_\Theta(\mathbf{b}, \mathbf{a}, h) : \mathbb{T}\mathrm{r}_n(\Theta^1(\mathbf{b})) \to \mathbb{T}\mathrm{r}_n(\Theta^1(\mathbf{b}))$

$$\mathbb{X}_\Theta(\mathbf{b},\mathbf{a},h)\Big( \begin{tikzcd} n \arrow[r, bend left, "\Theta(\mathbf{b})"] \arrow[r, bend right, "\mathbf{i}_n"'] \arrow[r, phantom, "\Uparrow v"] & n \end{tikzcd} \Big) = \begin{tikzcd}[column sep=large] n \arrow[rrr, bend left=70, "\Theta(\mathbf{b})"] \arrow[rrr, bend right=70, "\mathbf{i}_n"'] \arrow[r, bend left, "\Theta(\mathbf{a})"] \arrow[r, bend right, "\Theta(\mathbf{a})"'] \arrow[r, phantom, "\Uparrow Id"] & n \arrow[r, bend left, "\Theta(\mathbf{b})"] \arrow[r, bend right, "\mathbf{i}_n"'] \arrow[r, phantom, "\Uparrow v"] \arrow[r, phantom, shift left=10, "\Uparrow {[\mathbf{a},\mathbf{b},\mathbf{a}^{-1},h]}"] \arrow[r, phantom, shift right=10, "\Uparrow {[\mathbf{a},1,\mathbf{a}^{-1},1]^{-1}}"] & n \arrow[r, bend left, "\Theta(\mathbf{a}^{-1})"] \arrow[r, bend right, "\Theta(\mathbf{a}^{-1})"'] \arrow[r, phantom, "\Uparrow Id"] & n \end{tikzcd}$$

where $\Theta(\mathbf{b}) = \Theta^1_{\star,\star}(\mathbf{b})$ and $[\mathbf{a}, \mathbf{b}, \mathbf{a}^{-1}, h]$ is a composition of the natural morphism $\Theta(\mathbf{a}) \diamond \Theta(\mathbf{b}) \diamond \Theta(\mathbf{a}^{-1}) \to \Theta(\mathbf{a}\mathbf{b}\mathbf{a}^{-1})$ and the action $h\cdot : \Theta(\mathbf{a}\mathbf{b}\mathbf{a}^{-1}) \to \Theta(\mathbf{b})$. Let us write a matrix of vector spaces $\Theta^1_{\star,\star}(\mathbf{a}) = (U_{i,j}(\mathbf{a}))$. Its dimension is the permutation matrix of some permutation $\sigma$, e.g., $U_{i,j}(\mathbf{a}) \neq 0$ if and only if $j = \sigma(i)$. Now the natural map $\mathbf{i}_n \to \Theta(\mathbf{a}) \diamond \Theta(\mathbf{a}^{-1})$ is given by a collection of elements $x_i(\mathbf{a}) \in U_{i,\sigma(i)}(\mathbf{a})$, $y_i(\mathbf{a}) \in U_{\sigma(i),i}(\mathbf{a}^{-1})$ in a way that

$$\mathbb{K} = \mathbf{i}_{n\ i,i} \ni 1_i \mapsto x_i(\mathbf{a}) \otimes y_i(\mathbf{a}) \in U_{i,\sigma(i)}(\mathbf{a}) \otimes U_{\sigma(i),i}(\mathbf{a}^{-1}).$$

Now we can write the key map in an elementary way:

$$\mathbb{X}_\Theta(\mathbf{b},\mathbf{a},h)(\sum_i b_i) = h \cdot (\sum_i x_i(\mathbf{a}) \otimes b_{\sigma(i)} \otimes y_i(\mathbf{a})) \quad \text{where} \quad \sum_i b_i \in \mathbb{T}\mathrm{r}_n(\Theta^1_{\star.\star}(\mathbf{b})) = \oplus_i U_{i,i}(\mathbf{b}).$$

Its trace is the 2-character value

$$\mathfrak{X}_\Theta(\mathbf{b},\mathbf{a},h) \coloneqq \mathrm{Tr}(\mathbb{X}_\Theta(\mathbf{b},\mathbf{a},h)).$$

Let $\mathbb{G}$ be the set of all triples $(\mathbf{a}, \mathbf{b}, h) \in G \times G \times H$ such that $\partial(h)\mathbf{a}\mathbf{b} = \mathbf{b}\mathbf{a}$. The group $G$ acts on the set $\mathbb{G}$ by conjugation.

**Proposition 5.1.** *For any 2-representation $\Theta$ the function $\mathfrak{X}_\Theta : \mathbb{G} \to \mathbb{K}$ is constant on $G$-orbits. If $\Psi$ is another 2-representation, then*

$$\mathfrak{X}_{\Psi \boxtimes \Theta}(\mathbf{b},\mathbf{a},h) \coloneqq \mathfrak{X}_\Psi(\mathbf{b},\mathbf{a},h) \cdot \mathfrak{X}_\Theta(\mathbf{b},\mathbf{a},h).$$

*Proof.* Let us prove the first statement, i.e., that $\mathfrak{X}_\Theta({}^{\mathbf g}\mathbf b, {}^{\mathbf g}\mathbf a, {}^{\mathbf g}h) = \mathfrak{X}_\Theta(\mathbf b, \mathbf a, h)$ for all $\mathbf g \in G$. We have a natural "conjugation" linear map $\Gamma_{\mathbf g} : \mathbb{T}r_n(\Theta^1(\mathbf b)) \to \mathbb{T}r_n(\Theta^1({}^{\mathbf g}\mathbf b))$ given by the formula

$$\Gamma_{\mathbf g}\Bigg( \begin{tikzcd} n \arrow[r, bend left, "\Theta^1(\mathbf b)"] \arrow[r, bend right, "\mathbf i_n"'] \arrow[r, phantom, "\Uparrow v"] & n \end{tikzcd} \Bigg) = \begin{tikzcd} n \arrow[r, bend left, "\Theta^1(\mathbf g)"] \arrow[r, bend right, "\Theta^1(\mathbf g)"'] \arrow[r, phantom, "\Uparrow Id"] \arrow[rrr, bend left=60, "\Theta^1(\mathbf{gbg}^{-1})"] \arrow[rrr, bend right=60, "\mathbf i_n"'] \arrow[rrr, phantom, bend left=40, "\Uparrow \Theta^2_{\mathbf{gbg}^{-1},1}"] \arrow[rrr, phantom, bend right=40, "\Uparrow (\Theta^2_{1,1})^{-1}"] & n \arrow[r, bend left, "\Theta^1(\mathbf b)"] \arrow[r, bend right, "\mathbf i_n"'] \arrow[r, phantom, "\Uparrow v"] & n \arrow[r, bend left, "\Theta^1(\mathbf g^{-1})"] \arrow[r, bend right, "\Theta^1(\mathbf g^{-1})"'] \arrow[r, phantom, "\Uparrow Id"] & n \end{tikzcd}$$

It is straightforward to observe that

$$\mathbb{X}_\Theta({}^{\mathbf g}\mathbf b, {}^{\mathbf g}\mathbf a, {}^{\mathbf g}h) = \Gamma_{\mathbf g}\mathbb{X}_\Theta(\mathbf b, \mathbf a, h)\Gamma_{\mathbf g}^{-1},$$

hence, the linear maps $\mathbb{X}_\Theta({}^{\mathbf g}\mathbf b, {}^{\mathbf g}\mathbf a, {}^{\mathbf g}h)$ and $\mathbb{X}_\Theta(\mathbf b, \mathbf a, h)$ have the same trace. Let us prove the second statement now. Writing matrices of vector spaces as $\Theta^1_{\star,\star}(\mathbf b) = (U_{i,j}(\mathbf b))$ and $\Psi^1_{\star,\star}(\mathbf b) = (V_{i,j}(\mathbf b))$, we observe that $\mathbb{T}\mathrm{r}_{nm}(\Theta \boxtimes \Psi)^1_{\star,\star}(\mathbf b) \cong \sum_{i,t} U_{i,i}(\mathbf b) \otimes V_{t,t}(\mathbf b) \cong (\sum_i U_{i,i}(\mathbf b)) \otimes (\sum_t V_{t,t}(\mathbf b)) \cong \mathbb{T}\mathrm{r}_n \Theta^1_{\star,\star}(\mathbf b) \otimes \mathbb{T}\mathrm{r}_m \Psi^1_{\star,\star}(\mathbf b)$. Identifying the 2-traces under these maps, we can observe that $\mathbb{X}_{\Theta\boxtimes\Psi}(\mathbf b, \mathbf a, h) = \mathbb{X}_\Theta(\mathbf b, \mathbf a, h) \otimes \mathbb{X}_\Psi(\mathbf b, \mathbf a, h)$ that implies the second statement. □

A character table of a finite group has rows and columns. Usually one thinks of columns as characters, yet it is often instructive to think of rows as characters. Applying this way of thinking to the 2-characters we can use Proposition 5.1 to conclude that a $G$-conjugacy class of triples $(\mathbf a, \mathbf b, h)$ with $\partial(h)\mathbf{ab} = \mathbf{ba}$ determines a ring homomorphism

$$\mathfrak{X}(\mathbf b, \mathbf a, h) : \mathbb{B}_{\mathbb Z}(\mathcal K) \to \mathbb K, \quad [\Theta] \mapsto \mathfrak{X}_\Theta(\mathbf b, \mathbf a, h).$$

It can be extended by $\mathbb K$-linearity to a $\mathbb K$-algebra homomorphism $\mathfrak{X}(\mathbf b, \mathbf a, h) : \mathbb{B}_{\mathbb K}(\mathcal K) \to \mathbb K$. Both versions of $\mathfrak X$ should be called *a Ganter-Kapranov 2-character*. In the finite case (i.e., under assumptions of Proposition 4.3) the Ganter-Kapranov 2-character must be one of the marks. Which one?

**Theorem 5.2.** *In the notations above, let $P$ be the subgroup of $\pi_1(\mathcal K)$ generated by $\bar{\mathbf a}$ and $\bar{\mathbf b}$.*
*Let $\alpha \coloneqq \mathfrak{X}(\mathbf b, \mathbf a, h)$ considered as a group homomorphism $2\text{-Rep}^1(\mathcal K_P) \to \mathbb K^\times$. If the order of $\pi_1(\mathcal K)$ is finite and invertible in the field $\mathbb K$, then*

$$\mathfrak{X}(\mathbf b, \mathbf a, h) = f_P^\alpha.$$

*Proof.* Let us compute both parts on an induced 2-representation. Let $Q$ be a subgroup of $\pi_1(\mathcal K)$, $\widehat Q$ its inverse image in $G$, $\Theta = \Theta^\circ_A \in 2\text{-Rep}^1(\mathcal K_Q)$ that comes from an action of $\mathcal K_Q$ on a full matrix algebra $A = \mathrm{End}_{\mathbb K}(M)$. Let us choose a left transversal $T$ to $\widehat Q$ in $G$ so that $\widetilde\Theta \coloneqq \Theta^\circ_A \uparrow^{\widetilde{\mathcal K}}_{\mathcal K_Q} = \Theta^\circ_{\widetilde A}$. where $\widetilde A = \oplus_{\mathbf t \in T}(\mathbf t, A)$.

Let $n = |T|$. The contribution to $\mathbb{T}\mathrm{r}_n(\widetilde{\Theta}(\mathbf{b}))$ comes from those $\mathbf{t}$ that $\mathbf{bt} \in \mathbf{t}\widehat{Q}$ (equivalently $\mathbf{t}^{-1}\mathbf{bt} \in \widehat{Q}$):

$$\mathbb{T}\mathrm{r}_n(\widetilde{\Theta}(\mathbf{b})) = \bigoplus_{\mathbf{t}\in T} \hom_{\widetilde{A}}((\mathbf{t}, M), (\mathbf{bt}, M)) \cong \bigoplus_{\mathbf{t}\in T,\ \mathbf{t}^{-1}\mathbf{bt}\in\widehat{Q}} \mathbb{K}$$

where $(\mathbf{x}, M)$ is a simple module for $(\mathbf{x}, A)$. Observe that $(\mathbf{bt}, M) = (\mathbf{t}, M)^{[\mathbf{b}]}$ and the condition $\mathbf{t}^{-1}\mathbf{bt} \in \widehat{Q}$ is equivalent to $\widetilde{A}$-modules $(\mathbf{bt}, M)$ and $(\mathbf{t}, M)$ being isomorphic. The linear map $\mathbb{X}_\Theta(\mathbf{b}, \mathbf{a}, h)$ goes via the matrix of vector spaces $\widetilde{\Theta}(\mathbf{a}) \diamond \widetilde{\Theta}(\mathbf{b}) \diamond \widetilde{\Theta}(\mathbf{a}^{-1})$ whose entries on the main diagonal are of the form

$$U_{i,j}(\mathbf{a}) \otimes U_{j,j}(\mathbf{b}) \otimes U_{j,i}(\mathbf{a}^{-1}) = \hom_{\widetilde{A}}((\mathbf{bt}, M), (\mathbf{abt}, M)) \otimes \hom_{\widetilde{A}}((\mathbf{t}, M), (\mathbf{bt}, M)) \otimes \hom_{\widetilde{A}}((\mathbf{a}^{-1}\mathbf{t}, M), (\mathbf{t}, M)).$$

For this entry to be nonzero we need the three elements $(\mathbf{a}^{-1}\mathbf{t})^{-1}\mathbf{t} = \mathbf{t}^{-1}\mathbf{at}$, $\mathbf{t}^{-1}\mathbf{bt}$ and $(\mathbf{bt})^{-1}\mathbf{abt} = \mathbf{t}^{-1}\mathbf{b}^{-1}\mathbf{abt}$ to be in $\widehat{Q}$. This is equivalent to $\mathbf{t}^{-1}\widehat{P}\mathbf{t} \subseteq \widehat{Q}$.

It remains to notice that the contribution from each such $\mathbf{t}$ is value of $\mathfrak{X}(\mathbf{b}, \mathbf{a}, h)$ on the pull-back (under $\gamma_{\mathbf{t}}$) of the degree one 2-representation $\Theta$. The theorem follows immediately by examining the formula for $f_P^\alpha$. □

## 6. Shapiro isomorphism

Let $G$ be a group, $Q \leq G$ its subgroup, $M$ a $\mathbb{Z}Q$-module. Shapiro's lemma [14] asserts isomorphisms in homology and cohomology:

$$H^*(G, \mathrm{Coind}_Q^G(M)) \cong H^*(Q, M), \quad H_*(G, \mathrm{Ind}_Q^G(M)) \cong H_*(Q, M).$$

The standard proof goes via a quasiisomorphism of the corresponding complexes. It does not give an explicit formula that we require for cohomology. Hence, we supply an explicit chain homotopy

$$\psi : C^n(Q, M) \to C^n(G, \mathrm{Coind}_Q^G(M)).$$

Choose a right transversal $T = \{\mathbf{t}_1, \mathbf{t}_2 \ldots\}_j$ to $Q$ in $G$ such that $\mathbf{t}_1 = 1_G$. The coinduced module $\mathrm{Coind}_Q^G(M)$ is the set of all $Q$-equivariant functions $f : G \to M$. Such a function is uniquely determined by its values on $T$. The right transversal allows us to identify the coinduced module $\mathrm{Coind}_Q^G(M)$ with the set of all functions $f : T \to M$. The cochains $C^n(Q, M)$ are also functions $\mu : Q^n \to M$. Given elements $\mathbf{g}_1, \ldots, \mathbf{g}_n \in G$ and $\mathbf{t} \in T$, there exist elements $\mathbf{h}_1, \ldots, \mathbf{h}_n \in Q$ and $\mathbf{s}_0 = \mathbf{t}, \mathbf{s}_1, \ldots, \mathbf{s}_n \in T$ uniquely determined by the following equations:

$$\mathbf{s}_0\mathbf{g}_1 \cdots \mathbf{g}_n = \mathbf{h}_1\mathbf{s}_1\mathbf{g}_2 \cdots \mathbf{g}_n = \ldots = \mathbf{h}_1 \cdots \mathbf{h}_k\mathbf{s}_k\mathbf{g}_{k+1} \cdots \mathbf{g}_n = \ldots = \mathbf{h}_1 \cdots \mathbf{h}_{n-1}\mathbf{s}_{n-1}\mathbf{g}_n = \mathbf{h}_1 \cdots \mathbf{h}_n\mathbf{s}_n.$$

We use these elements to define $\psi$ on a cochain $\mu \in C^n(Q, M)$:

$$\psi(\mu)(\mathbf{g}_1, \ldots, \mathbf{g}_n)(\mathbf{t}) \coloneqq \mu(\mathbf{h}_1, \ldots, \mathbf{h}_n).$$

In the opposite direction we define a map for arbitrary elements $\mathbf{h}_1, \ldots, \mathbf{h}_n \in Q$:

$$\phi : C^n(G, \mathrm{Coind}_Q^G(M)) \to C^n(Q, M), \quad \phi(\theta)(\mathbf{h}_1, \ldots, \mathbf{h}_n) = \theta(\mathbf{h}_1, \ldots, \mathbf{h}_n)(1_G).$$

We are ready for the main result of this section.

**Theorem 6.1.** *Let $Q \leq G$ be groups, $M$ a $\mathbb{Z}Q$-module. The above defined maps $\phi$ and $\psi$ are isomorphisms of the cochain complexes $C^*(Q, M)$ and $C^*(G, \mathrm{Coind}_Q^G(M))$ in the homotopic category.*

*Proof.* Observe that for $\mathbf{g}_1, \ldots, \mathbf{g}_n \in Q$ and $\mathbf{t} = 1$ we get $\mathbf{h}_j = \mathbf{g}_j$. Hence,

$$\phi(\psi(\mu))(\mathbf{g}_1, \ldots, \mathbf{g}_n) = \psi(\mu)(\mathbf{g}_1, \ldots, \mathbf{g}_n)(1_G) = \mu(\mathbf{g}_1, \ldots, \mathbf{g}_n)$$

proving that $\phi \circ \psi$ is equal to the identity. In the opposite direction, $\psi \circ \phi$ is only homotopic to the identity:

$$\begin{tikzcd}
\cdots \arrow[r] & C^{n-1}(P, \mathrm{Coind}_Q^G(M)) \arrow[r, "d^{n-1}"] \arrow[dl] \arrow[d, shift right, "\mathbb{1}"'] \arrow[d, shift left, "\phi\circ\psi"] & C^{n}(P, \mathrm{Coind}_Q^G(M)) \arrow[r, "d^{n}"] \arrow[dl, "\varpi^{n-1}"'] \arrow[d, shift right, "\mathbb{1}"'] \arrow[d, shift left, "\phi\circ\psi"] & C^{n+1}(P, \mathrm{Coind}_Q^G(M)) \arrow[r] \arrow[dl, "\varpi^{n}"'] \arrow[d, shift right, "\mathbb{1}"'] \arrow[d, shift left, "\phi\circ\psi"] & \cdots \arrow[dl] \\
\cdots \arrow[r] & C^{n-1}(P, \mathrm{Coind}_Q^G(M)) \arrow[r, "d^{n-1}"] & C^{n}(P, \mathrm{Coind}_Q^G(M)) \arrow[r, "d^{n}"] & C^{n+1}(P, \mathrm{Coind}_Q^G(M)) \arrow[r] & \cdots
\end{tikzcd}$$

where the homotopy $\varpi$ is define by

$$\varpi^n(\theta)(\mathbf{g}_1, \ldots, \mathbf{g}_n)(\mathbf{t}) = \sum_{j=0}^{n} (-1)^{j+1} \theta(\mathbf{h}_1, \ldots, \mathbf{h}_j, \mathbf{s}_j, \mathbf{g}_{j+1}, \ldots, \mathbf{g}_n)(1).$$

Let us verify that $\psi \circ \phi - \mathbb{1} = h^n \varpi^n + d^{n-1} \varpi^{n-1}$. Let us first examine the left hand side of this equality:

$$(\psi \circ \phi - \mathbb{1})(\theta)(\mathbf{g}_1, \ldots, \mathbf{g}_n)(\mathbf{t}) = \psi \circ \phi(\theta)(\mathbf{g}_1, \ldots, \mathbf{g}_n)(\mathbf{t}) - \theta(\mathbf{g}_1, \ldots, \mathbf{g}_n)(\mathbf{t}) = \theta(\mathbf{h}_1, \ldots, \mathbf{h}_n)(1) - \theta(\mathbf{g}_1, \ldots, \mathbf{g}_n)(\mathbf{t}).$$

Now we scrutinise the first term of the right hand side. It is useful to pay attention which $\mathbf{s}_j$ appears in terms of the final expression because it tells you from which term of the second expression it originates. We label the lines to help observe the cancellations:

$$
\begin{aligned}
\varpi^n(d^n(\theta))(\mathbf{g}_1,\ldots,\mathbf{g}_n)(\mathbf{t}) = \sum_{j=0}^{n}(-1)^{j+1}d^n(\theta)(\mathbf{h}_1,\ldots,\mathbf{h}_j,\mathbf{s}_j,\mathbf{g}_{j+1},\ldots,\mathbf{g}_n)(1) = \\
-\,\theta(\mathbf{g}_1,\ldots,\mathbf{g}_n)(\mathbf{s}_0) \qquad (1)\\
+\sum_{j=1}^{n}(-1)^{j+1}\theta(\mathbf{h}_2,\ldots,\mathbf{h}_j,\mathbf{s}_j,\mathbf{g}_{j+1},\ldots,\mathbf{g}_n)(\mathbf{h}_1)\;+ \qquad (2)\\
+\sum_{j=2}^{n}\sum_{k=1}^{j-1}(-1)^{j+k+1}\theta(\ldots\mathbf{h}_{k-1},\mathbf{h}_k\mathbf{h}_{k+1},\mathbf{h}_{k+2},\ldots,\mathbf{h}_j,\mathbf{s}_j,\mathbf{g}_j\ldots)(1) \qquad (3)\\
-\sum_{j=1}^{n}\theta(\mathbf{h}_1,\ldots,\mathbf{h}_{j-1},\mathbf{h}_j\mathbf{s}_j,\mathbf{g}_{j+1},\ldots,\mathbf{g}_n)(1) \qquad (4)\\
+\sum_{j=0}^{n-1}\theta(\mathbf{h}_1,\ldots,\mathbf{h}_{j-1},\mathbf{s}_j\mathbf{g}_{j+1},\mathbf{g}_{j+1},\ldots,\mathbf{g}_n)(1) \qquad (5)\\
+\sum_{j=0}^{n-2}\sum_{k=j+1}^{n-1}(-1)^{j+k}\theta(\ldots\mathbf{h}_j,\mathbf{s}_j,\mathbf{g}_{j+1},\ldots,\mathbf{g}_{k-1},\mathbf{g}_k\mathbf{g}_{k+1},\mathbf{g}_{k+2}\ldots)(1) \qquad (6)\\
+\sum_{j=0}^{n-1}(-1)^{j+n}\theta(\mathbf{h}_1,\ldots,\mathbf{h}_j,\mathbf{s}_j,\mathbf{g}_{j+1},\ldots,\mathbf{g}_{n-1})(1) \qquad (7)\\
+\,\theta(\mathbf{h}_1,\ldots,\mathbf{h}_n)(1). \qquad (8)
\end{aligned}
$$

Lines (1) and (8) contribute to the left hand side. Lines (4) and (5) cancel because $\mathbf{s}_j\mathbf{g}_{j+1} = \mathbf{h}_{j+1}\mathbf{s}_{j+1}$. The remaining lines cancel with the second term (line labels correspond to their cancelling counterparts):

$$
\begin{aligned}
& d^{n-1}(\varpi^{n-1}(\theta))(\mathbf{g}_1,\ldots,\mathbf{g}_n)(\mathbf{t}) = \\
& \varpi^{n-1}(\theta)(\mathbf{g}_2,\ldots,\mathbf{g}_n)(\mathbf{h}_1\mathbf{s}_1) + \sum_{k=1}^{n-1}(-1)^k\varpi^{n-1}(\theta)(\mathbf{g}_1,\ldots,\mathbf{g}_k\mathbf{g}_{k+1}\ldots)(\mathbf{t}) + (-1)^n\varpi^{n-1}(\theta)(\mathbf{g}_1,\ldots,\mathbf{g}_{n-1})(\mathbf{t})\\
& = \sum_{j=1}^{n}(-1)^j\theta(\mathbf{h}_2,\ldots,\mathbf{h}_j,\mathbf{s}_j,\mathbf{g}_{j+1},\ldots,\mathbf{g}_n)(\mathbf{h}_1) \qquad (2)\\
& + \sum_{k=1}^{n-1}\sum_{j=0}^{k-1}(-1)^{k+j+1}\theta(\ldots\mathbf{h}_j,\mathbf{s}_j,\mathbf{g}_{j+1},\ldots,\mathbf{g}_{k-1},\mathbf{g}_k\mathbf{g}_{k+1},\mathbf{g}_{k+2}\ldots)(1) \qquad (6)\\
& + \sum_{k=1}^{n-1}\sum_{j=k+1}^{n-1}(-1)^{k+j}\theta(\ldots\mathbf{h}_{k-1},\mathbf{h}_k\mathbf{h}_{k+1},\mathbf{h}_{k+2},\ldots,\mathbf{h}_j,\mathbf{s}_j,\mathbf{g}_{j+1},\ldots,\mathbf{g}_n)(1) \qquad (3)\\
& + \sum_{j=0}^{n-1}(-1)^{n+j+1}\theta(\mathbf{h}_1,\ldots,\mathbf{h}_j,\mathbf{s}_j,\mathbf{g}_{j+1},\ldots,\mathbf{g}_{n-1})(1). \qquad (7)
\end{aligned}
$$

□

## 7. Osorno Formula

In this section we investigate the special case of trivial $H$. Thus, $G = \pi_1(\mathcal{K})$ is a finite group $G = \pi_1(\mathcal{K})$. We will write $G$ for $\mathcal{K}$ where appropriate, e.g., 2-Rep$(G)$ = 2-Rep$(\mathcal{K})$ etc. The degree one 2-representations of $G$ are in bijection with elements of the Schur multiplier over $\mathbb{K}$:

**Proposition 7.1.** *(cf. [7, 5.3]) The group of degree one 2-representations* $(2\text{-Rep}_1(G), \boxtimes)$ *(see Lemma 3.6) is isomorphic to* $H^2(G, \mathbb{K}^\times)$ *where the multiplicative group* $\mathbb{K}^\times$ *is a trivial* $\mathbb{Z}G$*-module.*

*Proof.* Let $\Theta$ be a degree one 2-representation of $G$. Lemma 3.3 attaches to $\Theta$ a unique $G$-Morita equivalence class of split simple algebras, whose representative $A$ satisfies $\Theta \cong \Theta_A$. The cohomology class $\{\Theta\} \in H^2(G, \mathbb{K}^\times)$ of the corresponding projective representation $G \to \mathrm{Aut}(A) \cong \mathrm{PGL}_n(\mathbb{K})$ defines a bijection [9, Theorem 4.3]

$$\langle \Theta, G \rangle \mapsto \{\Theta\}, \quad 2\text{-Rep}_1(G) \to H^2(G, \mathbb{K}^\times).$$

It is a group isomorphism because the tensor product of algebras correspond to the addition of cocycles. □

Since $H$ is trivial we drop $h$ from the notation for the Ganter-Kapranov 2-character: $\mathfrak{X}(\mathbf{b}, \mathbf{a}) \coloneqq \mathfrak{X}(\mathbf{b}, \mathbf{a}, 1)$. Let us compute its value on a degree one 2-representation:

**Theorem 7.2.** *Let* $\mathbf{a}, \mathbf{b} \in G$ *be commuting elements,* $\Theta$ *a degree one 2-representation of* $G$*,* $\mu \in Z^2(G, \mathbb{K}^\times)$ *a cocycle such that* $[\mu] = \{\Theta\}$*. Then*

$$\mathfrak{X}(\mathbf{b}, \mathbf{a})(\langle \Theta, G \rangle) = \mu(\mathbf{b}, \mathbf{a}^{-1})\mu(\mathbf{a}^{-1}, \mathbf{b})^{-1}.$$

*Proof.* Let $A$ be a split simple $G$-algebra such that $\Theta \cong \Theta_A$, $M$ a simple $A$-module. Let $\rho$ be the projective representation of $G$ on $M$ such that $\rho(1) = \mathrm{Id}_M$. The cocycle $\nu$ defined by the identity

$$\rho(\mathbf{gh}) = \nu(\mathbf{g}, \mathbf{h})\rho(\mathbf{g})\rho(\mathbf{h}) \quad \text{for all} \quad \mathbf{g}, \mathbf{h} \in G$$

satisfies $[\nu] = \{\Theta\}$. Then $\mu = \nu \cdot d\pi$ for some cochain $\pi \in C^1(G, \mathbb{K}^\times)$. One can use $\nu$ rather than $\mu$ on the right hand side:

$$\begin{aligned}\mu(\mathbf{b}, \mathbf{a}^{-1})\mu(\mathbf{a}^{-1}, \mathbf{b})^{-1} =& (\nu(\mathbf{b}, \mathbf{a}^{-1})\pi(\mathbf{b})\pi(\mathbf{ba}^{-1})^{-1}\pi(\mathbf{a}^{-1}))(\mu(\mathbf{a}^{-1}, \mathbf{b})\pi(\mathbf{a}^{-1})\pi(\mathbf{a}^{-1}\mathbf{b})^{-1}\pi(\mathbf{b}))^{-1} \\ =& \nu(\mathbf{b}, \mathbf{a}^{-1})\nu(\mathbf{a}^{-1}, \mathbf{b})^{-1}.\end{aligned}$$

The condition $\rho(1) = \mathrm{Id}_M$ makes the cocycle $\nu$ normalised and brings additional identities:

$$\nu(\mathbf{g}, 1) = \nu(1, \mathbf{g}) = 1, \quad \nu(\mathbf{g}, \mathbf{g}^{-1}) = \nu(\mathbf{g}^{-1}, \mathbf{g}), \quad \rho(\mathbf{g}^{-1})^{-1} = \nu(\mathbf{g}, \mathbf{g}^{-1})\rho(\mathbf{g}).$$

The linear map $\mathbb{X}_\Theta(\mathbf{b},\mathbf{a})$ operates on the one-dimensional space $\hom_A(M, M^{[\mathbf{b}]})$, a subspace of $\hom_{\mathbb{K}}(M,M)$ spanned by $\rho(\mathbf{b})$. More precisely,

$$\begin{aligned}\mathbb{X}_\Theta(\mathbf{b},\mathbf{a})(\rho(\mathbf{b})) =& \rho(\mathbf{a}^{-1})\rho(\mathbf{b})\rho(\mathbf{a}^{-1})^{-1} = \nu(\mathbf{a}^{-1},\mathbf{b})^{-1}\rho(\mathbf{a}^{-1}\mathbf{b})\nu(\mathbf{a},\mathbf{a}^{-1})\rho(\mathbf{a})\\ =& \nu(\mathbf{a}^{-1},\mathbf{b})^{-1}\nu(\mathbf{a}^{-1}\mathbf{b},\mathbf{a})^{-1}\nu(\mathbf{a},\mathbf{a}^{-1})\rho(\mathbf{b}).\end{aligned}$$

We can finish the proof using the cocycle condition and the fact that $\mathbf{a}^{-1}$ and $\mathbf{b}$ commute:

$$\begin{aligned}\nu(\mathbf{a}^{-1},\mathbf{b})^{-1}\nu(\mathbf{a},\mathbf{a}^{-1})\nu(\mathbf{b}\mathbf{a}^{-1},\mathbf{a})^{-1} =& \nu(\mathbf{a}^{-1},\mathbf{b})^{-1}\nu(\mathbf{a},\mathbf{a}^{-1})\nu(\mathbf{a}^{-1},\mathbf{a})^{-1}\nu(\mathbf{b},1)^{-1}\nu(\mathbf{b},\mathbf{a}^{-1})\\ =& \nu(\mathbf{b},\mathbf{a}^{-1})\nu(\mathbf{a}^{-1},\mathbf{b})^{-1}.\end{aligned}$$

□

Occasionally in the literature the opposite cocycle is associated to a projective representation: one can use $\nu(\mathbf{g},\mathbf{h})\rho(\mathbf{gh}) = \rho(\mathbf{g})\rho(\mathbf{h})$ instead. Then the formula for $\mathfrak{X}(\mathbf{b},\mathbf{a})(\langle\Theta,G\rangle)$ in Theorem 7.2 changes to its reciprocal. Other choices leading to the reciprocal are using right representations instead of left ones or using $\mathbf{a}^{-1}\mathbf{ba}$ in the definition of Ganter-Kapranov 2-character. We are ready to derive a formula for an irreducible 2-representation:

**Corollary 7.3.** *Let $\Theta$ be a degree one 2-representation of a subgroup $P \le G$, $\mu \in Z^2(P,\mathbb{K}^\times)$ a cocycle such that $\{\Theta\} = [\mu]$. Let $T$ be a right transversal to $P$ in $G$. If ${}^{\mathbf{t}}\mathbf{a} := \mathbf{tat}^{-1}$ then*

$$\mathfrak{X}(\mathbf{b},\mathbf{a})(\langle\Theta,P\rangle) = \sum_{\mathbf{t}\in T,\ {}^{\mathbf{t}}\mathbf{a},{}^{\mathbf{t}}\mathbf{b}\in P} \frac{\mu({}^{\mathbf{t}}\mathbf{b},({}^{\mathbf{t}}\mathbf{a})^{-1})}{\mu(({}^{\mathbf{t}}\mathbf{a})^{-1},{}^{\mathbf{t}}\mathbf{b})} = \sum_{\mathbf{t}\in T,\ {}^{\mathbf{t}}\mathbf{a},{}^{\mathbf{t}}\mathbf{b}\in P} \frac{\mu({}^{\mathbf{t}}\mathbf{b},({}^{\mathbf{t}}\mathbf{a})^{-1})\mu({}^{\mathbf{t}}\mathbf{a},({}^{\mathbf{t}}\mathbf{b})({}^{\mathbf{t}}\mathbf{a})^{-1})}{\mu({}^{\mathbf{t}}\mathbf{a},({}^{\mathbf{t}}\mathbf{a})^{-1})\mu(1,1)}.$$

*Proof.* If $\mathbf{g}$ and $\mathbf{h}$ commute, then the cocycle condition implies the following identity:

$$\mu(\mathbf{g},\mathbf{hg}^{-1}) = \mu(\mathbf{g},\mathbf{g}^{-1}\mathbf{h}) = \mu(\mathbf{g}^{-1},\mathbf{h})^{-1}\mu(\mathbf{g},\mathbf{g}^{-1})\mu(1,\mathbf{h}) = \mu(\mathbf{g}^{-1},\mathbf{h})^{-1}\mu(\mathbf{g},\mathbf{g}^{-1})\mu(1,1).$$

Using Theorem 7.2, Theorem 5.2 and the definition of the mark homomorphism we compute the character:

$$\begin{aligned}\mathfrak{X}(\mathbf{b},\mathbf{a})(\Theta,P) &= \frac{1}{|P|}\sum_{\mathbf{g}\in G,\ {}^{\mathbf{g}}\mathbf{b},{}^{\mathbf{g}}\mathbf{a}\in P} \mu({}^{\mathbf{g}}\mathbf{b},({}^{\mathbf{g}}\mathbf{a})^{-1})\mu(({}^{\mathbf{g}}\mathbf{a})^{-1},{}^{\mathbf{g}}\mathbf{b})^{-1}\\ &= \sum_{\mathbf{t}\in T,\ {}^{\mathbf{t}}\mathbf{b},{}^{\mathbf{t}}\mathbf{a}\in P} \mu({}^{\mathbf{t}}\mathbf{b},({}^{\mathbf{t}}\mathbf{a})^{-1})\mu(({}^{\mathbf{t}}\mathbf{a})^{-1},{}^{\mathbf{t}}\mathbf{b})^{-1}\\ &= \sum_{\mathbf{t}\in T,\ {}^{\mathbf{t}}\mathbf{a},{}^{\mathbf{t}}\mathbf{b}\in P} \frac{\mu({}^{\mathbf{t}}\mathbf{b},({}^{\mathbf{t}}\mathbf{a})^{-1})\mu({}^{\mathbf{t}}\mathbf{a},({}^{\mathbf{t}}\mathbf{b})({}^{\mathbf{t}}\mathbf{a})^{-1})}{\mu({}^{\mathbf{t}}\mathbf{a},({}^{\mathbf{t}}\mathbf{a})^{-1})\mu(1,1)}.\end{aligned}$$

□

Corollary 7.3 allows us to compute the value of the Ganter-Kapranov 2-character on any 2-representation in terms of its decorated $G$-set [9], i.e. a finite $G$-set $X$, decorated with a cocycle $\mu_x \in Z^2(G_x,\mathbb{K}^\times)$ at every point $x \in X$. An alternative data describing a representation is a cocycle on a permutation module [16, Proposition 1]. To describe we need a notation $(\mathbb{K}^\times)^X$ for the permutation $G$-module of all the function $X \to \mathbb{K}^\times$. Such a function $f$ is given by a collection of its values $(f(x)) = (\alpha_x)_{x\in X}$, i.e., non-zero field elements $\alpha_x \in \mathbb{K}^\times$. The action is left: $\mathbf{g} \cdot (\alpha_x) = (\alpha_{\mathbf{g}\cdot x})$. On the level functions it is given by $[\mathbf{g} \cdot f](x) = f(\mathbf{g}^{-1} \cdot x)$. of denotes $(\mathbb{C}^\times)^{|S|}$ as a $A$-module through the action of $A$ on $S$.

**Proposition 7.4.** *(cf. [16, Prop. 1] and [7, 5.4]) There is a one-to-one correspondence between equivalence classes of 2-representations of $G$ over $\mathbb{K}$ and pairs $(X, [\theta])$ where $X$ is a finite $G$-set and $[\theta] \in H^2(G, (\mathbb{K}^\times)^X)$.*

*Proof.* Theorem 3.7 associates to a 2-representation $\Theta$ a unique (up to conjugacy and an isomorphism) a collection $(P_i, \Phi_i)$ of pairs a subgroup $P_i$ and a degree one 2-representation $\Phi_i$ of $P_i$ so that

$$\Theta \cong \boxplus_i \Phi_i \uparrow_{P_i}^G .$$

Proposition 7.1 gives cohomology classes $\{\Phi_i\} \in H^2(P_i, \mathbb{K}^\times)$. The permutation module $(\mathbb{K}^\times)^{G/P_i}$ is naturally isomorphic to the coinduced module $\mathrm{Coind}_{P_i}^G(\mathbb{K}^\times)$, thus, we can use Shapiro isomorphism (see Theorem 6.1) to get unique cohomology classes $\psi(\{\Phi_i\}) \in H^2(G, (\mathbb{K}^\times)^{G/P_i})$. We have associated the set and the cohomology class

$$X \coloneqq \coprod_i G/P_i, \quad [\theta] \coloneqq \bigoplus_i \psi(\{\Phi_i\}) \in \bigoplus_i H^2(G, (\mathbb{K}^\times)^{G/P_i}) \cong H^2(G, (\mathbb{K}^\times)^X)$$

to $\Theta$. All these steps are reversible ◻

Given a finite $G$-set $X$, $x \in X$ and a cochain $\theta \in C^2(G, (\mathbb{K}^\times)^X)$, we write $\theta^x \in C^2(G, \mathbb{K}^\times)$ for the component cochains. We have $\theta(\mathbf{g}, \mathbf{h})(x) = \theta^x(\mathbf{g}, \mathbf{h})$ on the level of functions $X \to \mathbb{K}^\times$. We are ready to give our proof of Osorno Formula:

**Theorem 7.5.** *[16, Theorem 1] Let $\Theta$ be a 2-representation of $G$ that corresponds to a $G$-set $X$ and a cohomology class $[\theta]$ for some cochain $\theta \in Z^2(G, (\mathbb{K}^\times)^X)$. Then*

$$\mathfrak{X}_\Theta(\mathbf{b}, \mathbf{a}) = \sum_{x\in X,\ x=\mathbf{a}\cdot x=\mathbf{b}\cdot x} \frac{\theta^x(\mathbf{b}, \mathbf{a}^{-1})}{\theta^x(\mathbf{a}^{-1}, \mathbf{b})} = \sum_{x\in X,\ x=\mathbf{a}\cdot x=\mathbf{b}\cdot x} \frac{\theta^x(\mathbf{b}, \mathbf{a}^{-1})\theta^x(\mathbf{a}, \mathbf{b}\mathbf{a}^{-1})}{\theta^x(\mathbf{a}, \mathbf{a}^{-1})\theta^x(1,1)}$$

*for any commuting $\mathbf{a}, \mathbf{b} \in G$.*

*Proof.* The component $\theta^x$ is not a cocycle, in general. Yet for the terms in the formula it works as a cocycle: the restriction $\theta^x \mid_{<\mathbf{a},\mathbf{b}>}$ is a cocycle on $< \mathbf{a}, \mathbf{b} >$ since $x = \mathbf{a} \cdot x = \mathbf{b} \cdot x$. Thus, the second and the third expressions are equal.

Since $\mathfrak{X}_{\Theta\boxplus\Psi}(\mathbf{b}, \mathbf{a}) = \mathfrak{X}_\Theta(\mathbf{b}, \mathbf{a}) + \mathfrak{X}_\Psi(\mathbf{b}, \mathbf{a})$ and the second expression is additive on $G$-orbits. It suffices to prove the theorem under an assumption that $\Theta$ is irreducible. Without loss of generality $\Theta = \Psi \uparrow_P^G$ for a degree one 2-representation of some subgroup $P$ and $X = G/P$. Let $\mu \in Z^2(P, \mathbb{K}^\times)$ a cocycle such that $\{\Psi\} = [\mu]$. A right transversal $T$ (with $\mathbf{t}_0 = 1$) to $P$ in $G$ is in natural bijection with $X$ via $\mathbf{t} \mapsto \mathbf{t}^{-1}P$. We use $T$ and $\mu$ to decorate $X$ with cocycles:

$$\mu_{\mathbf{t}^{-1}P} \in Z^2(\mathbf{t}^{-1}P\mathbf{t}, \mathbb{K}^\times), \quad \mu_{\mathbf{t}^{-1}P}(\mathbf{g}, \mathbf{h}) \coloneqq \mu({}^{\mathbf{t}}\mathbf{g}, {}^{\mathbf{t}}\mathbf{h}).$$

By Corollary 7.3,

$$\mathfrak{X}_\Theta(\mathbf{b},\mathbf{a}) = \sum_{\mathbf{t}\in T,\ {}^{\mathbf{t}}\mathbf{a},{}^{\mathbf{t}}\mathbf{b}\in P} \frac{\mu({}^{\mathbf{t}}\mathbf{b},({}^{\mathbf{t}}\mathbf{a})^{-1})}{\mu(({}^{\mathbf{t}}\mathbf{a})^{-1},{}^{\mathbf{t}}\mathbf{b})} = \sum_{\mathbf{t}\in T,\ \mathbf{a}\mathbf{t}^{-1}P=\mathbf{b}\mathbf{t}^{-1}P=\mathbf{t}^{-1}P} \frac{\mu_{\mathbf{t}^{-1}P}(\mathbf{b},\mathbf{a}^{-1})}{\mu_{\mathbf{t}^{-1}P}(\mathbf{a}^{-1},\mathbf{b})} \ .$$

The cohomology classes of the cocycles $\mu_{\mathbf{t}^{-1}P}$ and $\theta$ are related via Shapiro isomorphisms with different subgroups: $[\mu_{\mathbf{t}^{-1}P}] = \phi_{\mathbf{t}^{-1}P\mathbf{t}}([\theta])$. Each term of the last sum depends only on cohomology class $[\mu_{\mathbf{t}^{-1}P}]$. Hence, we may assume that $\mu_{\mathbf{t}^{-1}P} = \phi_{\mathbf{t}^{-1}P\mathbf{t}}(\theta)$ without loss of generality. The condition $\mathbf{a},\mathbf{b}\in \mathbf{t}^{-1}P\mathbf{t}$ ensures that

$$\mu_{\mathbf{t}^{-1}P}(\mathbf{a},\mathbf{b}) = \phi_{\mathbf{t}^{-1}P\mathbf{t}}(\theta)(\mathbf{a},\mathbf{b}) = \theta(\mathbf{a},\mathbf{b})(\mathbf{t}^{-1}) = \theta^{\mathbf{t}^{-1}P}(\mathbf{a},\mathbf{b})$$

facilitating the last in the proof:

$$\mathfrak{X}_\Theta(\mathbf{b},\mathbf{a}) = \sum_{\mathbf{t}\in T,\ \mathbf{a}\mathbf{t}^{-1}P=\mathbf{b}\mathbf{t}^{-1}P=\mathbf{t}^{-1}P} \frac{\theta^{\mathbf{t}^{-1}P}(\mathbf{b},\mathbf{a}^{-1})}{\theta^{\mathbf{t}^{-1}P}(\mathbf{a}^{-1},\mathbf{b})} = \sum_{x\in X,\ \mathbf{a}\cdot x=\mathbf{b}\cdot x=x} \frac{\theta^{x}(\mathbf{b},\mathbf{a}^{-1})}{\theta^{x}(\mathbf{a}^{-1},\mathbf{b})} \ .$$

□

To facilitate further development we would like to formulate several questions (conjectures).

**Conjecture 1.** *Let $\mathcal{K} = (H \xrightarrow{\partial} G)$ be a crossed module with finite the fundamental group $\pi_1(\mathcal{K})$. Then there exists an Osorno formula for the value $\mathfrak{X}_\Theta(\mathbf{b},\mathbf{a},h)$ of the Ganter-Kapranov 2-character.*

In what cohomological terms would we expect the formula to play out? Group cohomology can be read off from the classifying space. Let $\mathcal{X}$ be a classifying space for $\mathcal{K}$. A 2-representation $\Theta$ of $\widetilde{\mathcal{K}}$ comes with a canonical $\pi_1(\mathcal{K})$-set $X$. The permutation representation $(\mathbb{K}^\times)^X$ of $\pi_1(\mathcal{K})$ defines a local system $\underline{(\mathbb{K}^\times)^X}$ on $\mathcal{X}$. We expect it to play a crucial role.

**Conjecture 2.** *If $\pi_1(\mathcal{K})$ is finite, then 2-representations of $\widetilde{\mathcal{K}}$ are classified by pair $(X,[\mu])$ where $X$ is a finite $\pi_1(\mathcal{K})$-set and $[\mu]$ is a cohomology class in $H^2(\mathcal{X},\underline{(\mathbb{K}^\times)^X})$ where $\mathcal{X}$ is a classifying space of $\mathcal{K}$.*

## 8. Examples of 2-representations

2-Representations are ubiquitous, yet mathematicians do not recognise them when they see them. We would like to give two well-known modern mathematical themes where they play crucial role, yet the statements are not formulated in the language of 2-representations.

The first story is Lusztig's conjectures [13]. Let $\mathfrak{g}$ be a simple finite dimensional complex Lie algebra, $\chi\in\mathfrak{g}^*$ its nilpotent character. The finite group $G$ attached to $\chi$ is the component group of the stabiliser of $\chi$ (or the fundamental group of the coadjoint orbit of $\chi$: they are naturally isomorphic).

Now there are two 2-representations of $G$ over $\mathbb{C}$ that appear in nature. The first 2-representation $\Theta_{\mathrm{Lie}}(\chi)$ comes from the action of $G$ on the semisimplification $U^0_\chi/\mathrm{Rad}(U^0_\chi)$ of the generic block $U^0_\chi$ of the reduced enveloping algebra. It is a 2-representation over an algebraically closed field $\mathbb{K}$ of prime characteristic $p$ that needs to be larger than the Coxeter number of $\mathfrak{g}$. For such $p$ we have a canonical isomorphism $H^2(G, \mathbb{K}^\times) \cong H^2(G, \mathbb{C}^\times)$ that leads to the 2-representation $\Theta_{\mathrm{Lie}}(\chi)$ over $\mathbb{C}$. It appears that this 2-representation may depend on characteristic $p$. Independence of $p$ can be established by the methods developed by Bezrukavnikov and Mirković [5], although there is no relevant result explicitly in the paper.

The second 2-representation $\Theta_{\mathrm{Coxeter}}(\chi)$ comes from geometry of the double cell of the Langlands dual affine Weyl group that corresponds to $\chi$ [12]. Bezrukavnikov and Ostrik extract a decorated $G$-set from this geometry [6], thus, a 2-representation of $G$ in our terminology. The following conjecture by Gunnells, Rose and Rumynin [9] is itself a reformulation of a conjecture by Lusztig [13]:

**Conjecture 3.** *(2-Lusztig Conjecture) The 2-representations $\Theta_{\mathrm{Lie}}(\chi)$ and $\Theta_{\mathrm{Coxeter}}(\chi)$ constructed above are equivalent. Moreover, their cohomology class $[\mu] \in H^2(G, (\mathbb{C}^\times)^X)$ is trivial.*

The second story is McKay's conjecture. One issue with proving it is that even if you know it holds for composition factors, it is not known how to deduce that the group itself satisfies it. Isaacs, Malle and Navarro suggest a stronger condition of McKay-goodness that is inherited by a group from its composition factors [10]. It appears that McKay goodness is a 2-representation-theoretic condition.

Let $H$ be a finite group, $p$ a fixed prime. Let $Z$ be the centre of $H$, $P$ a Sylow $p$-subgroup of $H$, $N$ the normaliser of $P$. The group $G$ of interest for us is the group of such automorphisms $\varphi : G \to G$ that $\varphi(P) = P$ and $\varphi|_Z = \mathrm{Id}_Z$. Let $\mathbb{C}H_{p'}$ be the direct summand of $\mathbb{C}H$ consisting of those matrix algebras whose size is not divisible $p$. It is a $\mathbb{C}Z$-algebra under the natural map $\mathbb{C}Z \to \mathbb{C}H \to \mathbb{C}H_{p'}$. It also has a $\mathbb{C}Z$-linear action of $G$. Likewise, there is a $\mathbb{C}Z$-linear action of $G$ on $\mathbb{C}N_{p'}$. For each character $\chi : Z \to \mathbb{C}^\times$ we get two 2-representations

$$\Theta_H(\chi) \coloneqq \Theta_{\mathbb{C}H_{p'} \otimes_{\mathbb{C}Z} \mathbb{C}(\chi)}, \quad \Theta_N(\chi) \coloneqq \Theta_{\mathbb{C}N_{p'} \otimes_{\mathbb{C}Z} \mathbb{C}(\chi)}.$$

**Conjecture 4.** *(2-McKay Conjecture) For each character $\chi$ the 2-representations $\Theta_H(\chi)$ and $\Theta_N(\chi)$ of $G$ are equivalent.*

Both conjectures would be solved if we could compute the values of marks on 2-representations. Indeed, two 2-representations are equivalent if and only if the values of all marks are equal as follows from Proposition 4.3. Thus, the most crucial question is to find efficient innovative methods for computing marks.

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